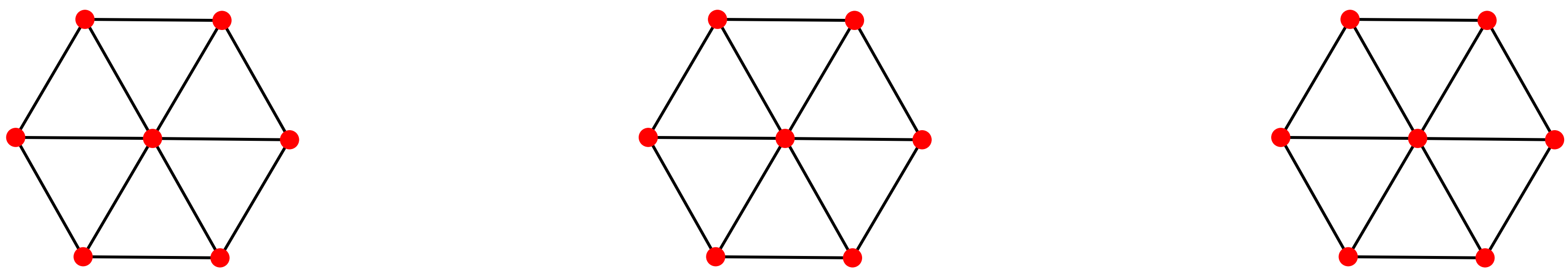
(Vertex) Colorings

Definition. A **coloring** of a graph G (with c colors) is a function $f : V(G) \to \{1, 2, \ldots, c\}$.

In other words, we assign colors to each of the vertices of G.

Definition. A **proper coloring** of G is a coloring of G such that no two adjacent vertices are labeled by the same color.

Example. W_6:



We can properly color W_6 with ____ colors and no fewer.

Of interest: What is the fewest colors necessary to properly color G?

The chromatic number of a graph

Definition. The minimum number of colors necessary to properly color a graph G is called the **chromatic number** of G, denoted $\chi(G) =$ "chi".

Example. $\chi(K_n) =$ ____

Proof. A proper coloring of K_n must use at least ____ colors, because every vertex is adjacent to every other vertex. With fewer than ____ colors, there would be two adjacent vertices colored the same. And indeed, placing a different color on each vertex is a proper coloring of K_n.

$\chi(G) = k$ is the same as:

1. There is a proper coloring of G with k colors. (Show it!)
2. There is no proper coloring of G with $k - 1$ colors. (Prove it!)

Chromatic numbers and subgraphs

Lemma C: If H is a subgraph of G, then $\chi(H) \leq \chi(G)$.

Proof. If $\chi(G) = k$, then there is a proper coloring of G using k colors.
Let the vertices of H inherit their coloring from G.
This gives a proper coloring of H using k colors.
In turn, this implies $\chi(H) \leq k$.

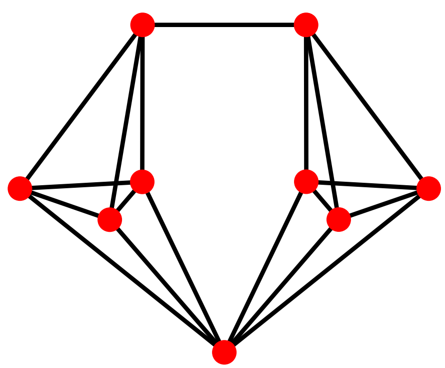
If G contains a **clique** of size k (subgraph isomorphic to K_k), then what can we say about $\chi(G)$?

Definition. The **clique number** $\omega(G)$ is the size of the largest complete graph contained in G.

Theorem. For any graph G, $\chi(G) \geq \omega(G)$.

Proof. Apply Lemma C to the subgraph of G isomorphic to $K_{\omega(G)}$.

Example. Calculate $\chi(G)$ for this graph G:



Critical graphs

How to prove $\chi(G) \geq k$?

One way: Find a (small) subgraph H of G that requires k colors.

Definition. A graph H is called **critical** if for every proper subgraph $J \subsetneq H$, then $\chi(J) < \chi(H)$.

Theorem 2.1.2: Every graph G contains a critical subgraph H such that $\chi(H) = \chi(G)$.

(Stupid) Proof. If G is critical, stop. Define $H = G$.

If not, then there exists a proper subgraph G_1 of G with ________.

If G_1 is critical, stop. Define $H = G_1$.

If not, then there exists a proper subgraph G_2 of G_1 with ______.

If G_2 is critical, stop. Define $H = G_2$.

If not, then there exists $\cdots$

Since G is finite, there will be some proper subgraph G_l of G_{l-1} such that G_l is critical and $\chi(G_l) = \chi(G_{l-1}) = \cdots = \chi(G)$.

Critical graphs

What do we know about critical graphs?

Thm 2.1.1: Every critical graph is connected.

Thm 2.1.3: If G is critical and $\chi(G) = 4$, then $\deg(v) \geq 3$ for all v.

Proof. Suppose not. Then there is some $v \in V(G)$ with $\deg(v) \leq 2$. Remove v from G to create H.

Similarly: If G is critical, then for all $v \in V(G)$, $\deg(v) \geq \chi(G) - 1$.

Bipartite graphs

Question. What is $\chi(C_n)$ when n is odd?

Answer.

Definition. A graph is called **bipartite** if $\chi(G) \leq 2$.

Example. $K_{m,n}$, $\square_n$, Trees

Thm 2.1.6: G is bipartite $\iff$ every cycle in G has even length.

$(\Rightarrow)$ Let G be bipartite. Assume that there is some cycle C of odd length contained in G . . .

Proof of Theorem 2.1.6

($\Leftarrow$) Suppose that every cycle in G has even length. We want to show that G is bipartite. Consider the case when G is connected.

Plan: Construct a coloring on G and prove that it is proper.

Choose some starting vertex x and color it blue. For every other vertex y, calculate the distance from y to x and then color y:

$$\begin{cases} \text{blue} & \text{if } d(x,y) \text{ is even.} \\ \text{red} & \text{if } d(x,y) \text{ is odd.} \end{cases}$$

Question: Is this a proper coloring of G?

If not, then there are two adjacent vertices v and w of the same color.

Claim 1: Their distance to the x is the same.

Claim 2: There exists an odd cycle in G.

This contradicts our hypothesis, so a 2-coloring exists; G is bipartite.

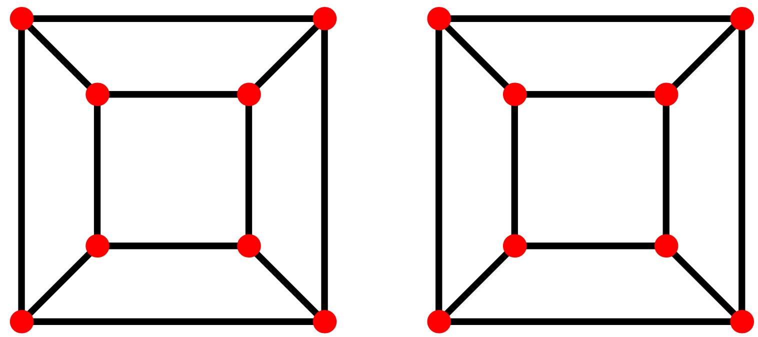
Edge Coloring

Parallel to the idea of vertex coloring is the idea of edge coloring.

Definition. An **edge coloring** of a graph G is a labeling of the edges of G with colors. [Technically, it is a function $f : E(G) \to \{1, 2, \ldots, l\}$.]

Definition. A **proper** edge coloring of G is an edge coloring of G such that no two *adjacent edges* are colored the same.

Example. Cube graph ($\square_3$):



We can properly edge color $\square_3$ with ____ colors and no fewer.

Definition. The minimum number of colors necessary to properly edge color a graph G is called the **edge chromatic number** of G, denoted $\chi'(G) =$ "chi prime".

Edge coloring theorems

Thm 2.2.1: For any graph G, $\chi'(G) \geq \Delta(G)$.

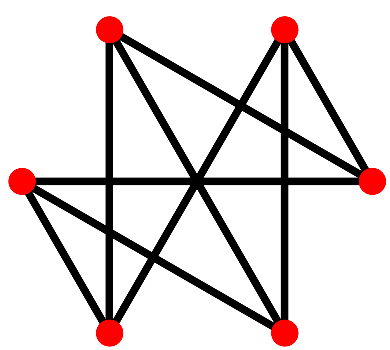
Thm 2.2.2: Vizing's Theorem:

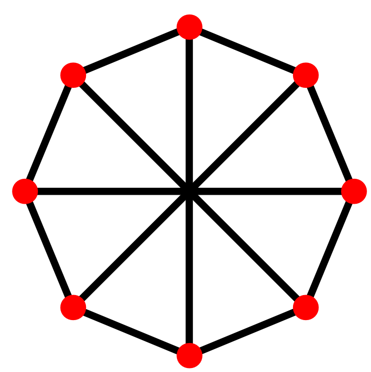
For any graph G, $\chi'(G)$ equals either $\Delta(G)$ or $\Delta(G) + 1$.

Proof. Hard. (See reference [24] if interested.)

Consequence: To determine $\chi'(G)$,

Fact: **Most** 3-regular graphs have edge chromatic number 3.



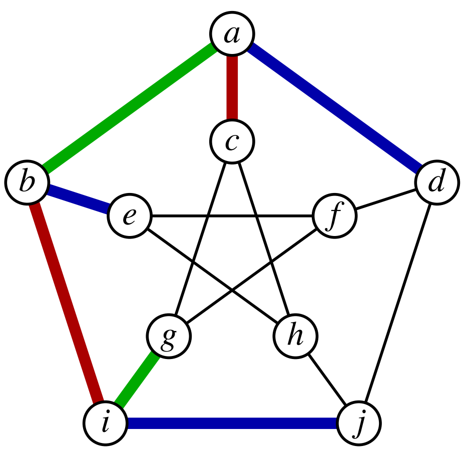


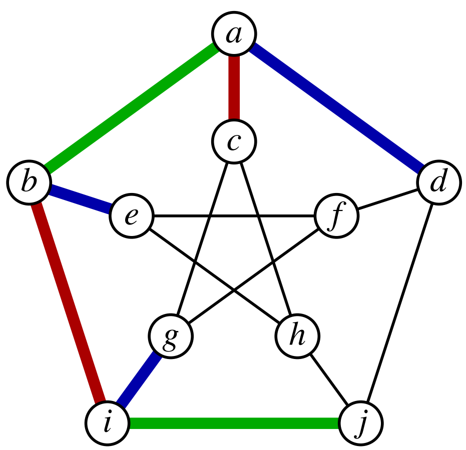
Snarks

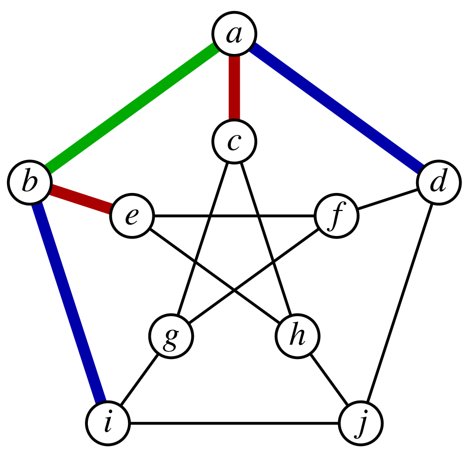
Definition. Another name for 3-regular is **cubic**.

Definition. A **snark** is a *bridgeless* cubic graph with edge chromatic number 4.

Example. The Petersen graph P is a snark. It is 3-regular. ✓
Let us prove that it can not be colored with three colors.
Assume you can color it with three colors. WLOG, assume ab, ac, ad.
Either **Case 1**: be and bi or **Case 2**: be and bi.
Either **Case 1a**: ig and ij or **Case 1b**: ig and ij.







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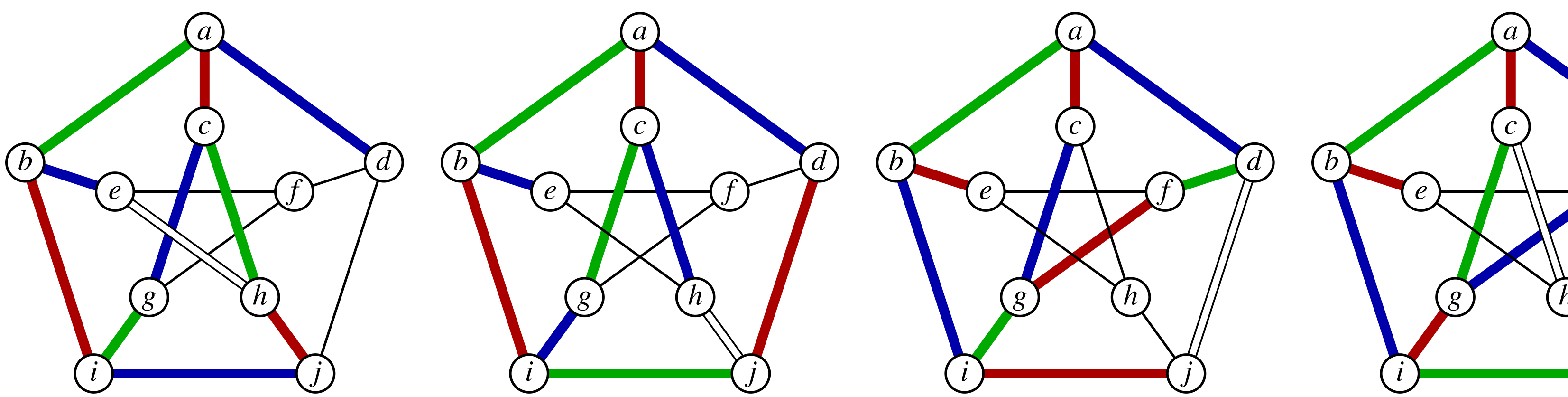
Either **Case 1**: be and bi or **Case 2**: be and bi.

Either **Case 1a**: ig and ij or **Case 1b**: ig and ij. **Cases 2a, 2b**



In all cases, it is not possible to edge color with 3 colors, so $\chi'(G) = 4$.

The edge chromatic number of complete graphs

Goal: Determine $\chi'(K_n)$ for all n.

Vertex Degree Analysis: The degree of every vertex in K_n is ____.

Vizing's theorem implies that $\chi'(K_n) =$____ or ____.

If $\chi'(K_n) =$____, then each vertex has an edge leaving of each color.

Question. How many red edges are there?

This is only an integer when:

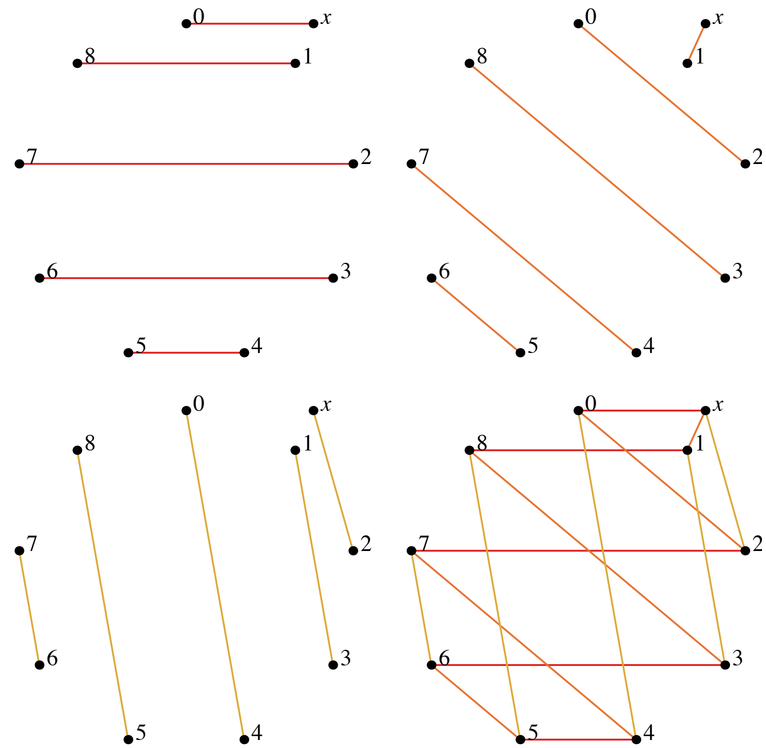
So, the best we can expect is that $\begin{cases} \chi'(K_{2n}) = \\ \chi'(K_{2n-1}) = \end{cases}$

The edge chromatic number of complete graphs

Thm 2.2.3: $\chi'(K_{2n}) = 2n - 1$.

Proof. We prove this using the *turning trick*.

Label the vertices of K_{2n}
$0, 1, \ldots, 2n-2, x$. Now,
Connect 0 with x,
Connect 1 with $2n-2$,
$\vdots$
Connect $n-1$ with n.



Now **turn** the inside edges.
And do it again. (and again, ...)

Each time, new edges are used.
This is because each of the edges is a different "circular length": vertices are at circ. distance 1, 3, 5, ..., 4, 2 from each other, and x is connected to a different vertex each time.

The edge chromatic number of complete graphs

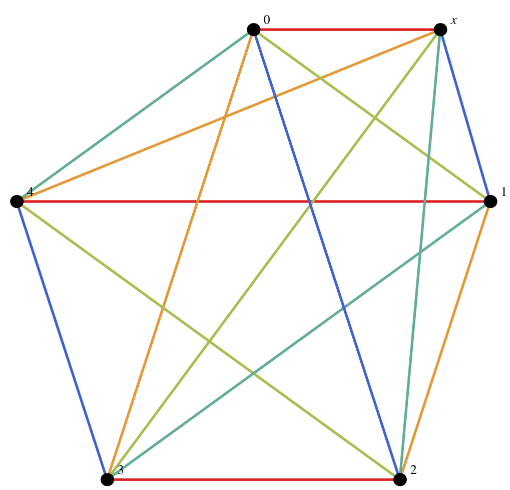
Theorem 2.2.4: $\chi'(K_{2n-1}) = 2n-1$.

This construction also gives a way to edge color K_{2n-1} with $2n-1$ colors—simply delete vertex x!

This is related to the area of combinatorial designs.

Question. Is it possible for six tennis players to play one match per day in a five-day tournament in such a way that each player plays each other player once?

Day 1	0x	14	23
Day 2	1x	20	34
Day 3	2x	31	40
Day 4	3x	42	01
Day 5	4x	03	12



Theorem 2.2.3 proves there is such a tournament for all even numbers.