(Vertex) Colorings

Definition. A **coloring** of a graph G (with c colors) is a function $f : V(G) \to \{1, 2, \ldots, c\}$.

In other words, we assign colors to each of the vertices of G.

(Vertex) Colorings

Definition. A **coloring** of a graph G (with c colors) is a function $f : V(G) \to \{1, 2, \ldots, c\}$.

In other words, we assign colors to each of the vertices of G.

Definition. A **proper coloring** of G is a coloring of G such that no two adjacent vertices are labeled by the same color.

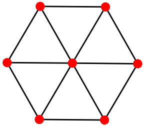
(Vertex) Colorings

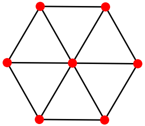
Definition. A **coloring** of a graph G (with c colors) is a function $f : V(G) \to \{1, 2, \ldots, c\}$.

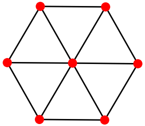
In other words, we assign colors to each of the vertices of G.

Definition. A **proper coloring** of G is a coloring of G such that no two adjacent vertices are labeled by the same color.

Example. W_6:







We can properly color W_6 with ____ colors and no fewer.

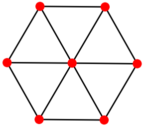
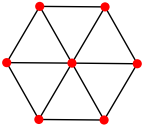
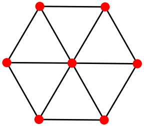
(Vertex) Colorings

Definition. A **coloring** of a graph G (with c colors) is a function $f : V(G) \to \{1, 2, \ldots, c\}$.

In other words, we assign colors to each of the vertices of G.

Definition. A **proper coloring** of G is a coloring of G such that no two adjacent vertices are labeled by the same color.

Example. W_6:

We can properly color W_6 with ____ colors and no fewer.

Of interest: What is the fewest colors necessary to properly color G?

The chromatic number of a graph

Definition. The minimum number of colors necessary to properly color a graph G is called the **chromatic number** of G, denoted $\chi(G) =$ "chi".

The chromatic number of a graph

Definition. The minimum number of colors necessary to properly color a graph G is called the **chromatic number** of G, denoted $\chi(G) =$ "chi".

Example. $\chi(K_n) =$ ____

The chromatic number of a graph

Definition. The minimum number of colors necessary to properly color a graph G is called the **chromatic number** of G, denoted $\chi(G) =$ "chi".

Example. $\chi(K_n) =$ ____

Proof. A proper coloring of K_n must use at least ____ colors, because every vertex is adjacent to every other vertex.

The chromatic number of a graph

Definition. The minimum number of colors necessary to properly color a graph G is called the **chromatic number** of G, denoted $\chi(G) =$ "chi".

Example. $\chi(K_n) =$ ____

Proof. A proper coloring of K_n must use at least ____ colors, because every vertex is adjacent to every other vertex. With fewer than ____ colors, there would be two adjacent vertices colored the same.

The chromatic number of a graph

Definition. The minimum number of colors necessary to properly color a graph G is called the **chromatic number** of G, denoted $\chi(G) =$ "chi".

Example. $\chi(K_n) =$ ____

Proof. A proper coloring of K_n must use at least ____ colors, because every vertex is adjacent to every other vertex. With fewer than ____ colors, there would be two adjacent vertices colored the same. And indeed, placing a different color on each vertex is a proper coloring of K_n.

The chromatic number of a graph

Definition. The minimum number of colors necessary to properly color a graph G is called the **chromatic number** of G, denoted $\chi(G) =$ "chi".

Example. $\chi(K_n) =$ ____

Proof. A proper coloring of K_n must use at least ____ colors, because every vertex is adjacent to every other vertex. With fewer than ____ colors, there would be two adjacent vertices colored the same. And indeed, placing a different color on each vertex is a proper coloring of K_n.

$\chi(G) = k$ is the same as:

The chromatic number of a graph

Definition. The minimum number of colors necessary to properly color a graph G is called the **chromatic number** of G, denoted $\chi(G) =$ "chi".

Example. $\chi(K_n) =$ ____

Proof. A proper coloring of K_n must use at least ____ colors, because every vertex is adjacent to every other vertex. With fewer than ____ colors, there would be two adjacent vertices colored the same. And indeed, placing a different color on each vertex is a proper coloring of K_n.

$\chi(G) = k$ is the same as:

1. There is a proper coloring of G with k colors.

The chromatic number of a graph

Definition. The minimum number of colors necessary to properly color a graph G is called the **chromatic number** of G, denoted $\chi(G) =$ "chi".

Example. $\chi(K_n) =$ ____

Proof. A proper coloring of K_n must use at least ____ colors, because every vertex is adjacent to every other vertex. With fewer than ____ colors, there would be two adjacent vertices colored the same. And indeed, placing a different color on each vertex is a proper coloring of K_n.

$\chi(G) = k$ is the same as:

1. There is a proper coloring of G with k colors.
2. There is no proper coloring of G with $k - 1$ colors.

The chromatic number of a graph

Definition. The minimum number of colors necessary to properly color a graph G is called the **chromatic number** of G, denoted $\chi(G) =$ "chi".

Example. $\chi(K_n) =$ ____

Proof. A proper coloring of K_n must use at least ____ colors, because every vertex is adjacent to every other vertex. With fewer than ____ colors, there would be two adjacent vertices colored the same. And indeed, placing a different color on each vertex is a proper coloring of K_n.

$\chi(G) = k$ is the same as:

1. There is a proper coloring of G with k colors. (Show it!)
2. There is no proper coloring of G with $k - 1$ colors. (Prove it!)

Chromatic numbers and subgraphs

Lemma C: If H is a subgraph of G, then $\chi(H) \leq \chi(G)$.

Chromatic numbers and subgraphs

Lemma C: If H is a subgraph of G, then $\chi(H) \leq \chi(G)$.

Proof. If $\chi(G) = k$, then

Chromatic numbers and subgraphs

Lemma C: If H is a subgraph of G, then $\chi(H) \leq \chi(G)$.

Proof. If $\chi(G) = k$, then there is a proper coloring of G using k colors.

Chromatic numbers and subgraphs

Lemma C: If H is a subgraph of G, then $\chi(H) \leq \chi(G)$.

Proof. If $\chi(G) = k$, then there is a proper coloring of G using k colors.
Let the vertices of H inherit their coloring from G.
This gives a proper coloring of H using k colors.

Chromatic numbers and subgraphs

Lemma C: If H is a subgraph of G, then $\chi(H) \leq \chi(G)$.

Proof. If $\chi(G) = k$, then there is a proper coloring of G using k colors.
Let the vertices of H inherit their coloring from G.
This gives a proper coloring of H using k colors.
In turn, this implies $\chi(H) \leq k$.

Chromatic numbers and subgraphs

Lemma C: If H is a subgraph of G, then $\chi(H) \leq \chi(G)$.

Proof. If $\chi(G) = k$, then there is a proper coloring of G using k colors.
Let the vertices of H inherit their coloring from G.
This gives a proper coloring of H using k colors.
In turn, this implies $\chi(H) \leq k$.

If G contains a **clique** of size k (subgraph isomorphic to K_k), then what can we say about $\chi(G)$?

Chromatic numbers and subgraphs

Lemma C: If H is a subgraph of G, then $\chi(H) \leq \chi(G)$.

Proof. If $\chi(G) = k$, then there is a proper coloring of G using k colors.
Let the vertices of H inherit their coloring from G.
This gives a proper coloring of H using k colors.
In turn, this implies $\chi(H) \leq k$.

If G contains a **clique** of size k (subgraph isomorphic to K_k),
then what can we say about $\chi(G)$?

Definition. The **clique number** $\omega(G)$ is the size of the largest complete graph contained in G.

Chromatic numbers and subgraphs

Lemma C: If H is a subgraph of G, then $\chi(H) \leq \chi(G)$.

Proof. If $\chi(G) = k$, then there is a proper coloring of G using k colors.
Let the vertices of H inherit their coloring from G.
This gives a proper coloring of H using k colors.
In turn, this implies $\chi(H) \leq k$.

If G contains a **clique** of size k (subgraph isomorphic to K_k), then what can we say about $\chi(G)$?

Definition. The **clique number** $\omega(G)$ is the size of the largest complete graph contained in G.

Theorem. For any graph G, $\chi(G) \geq \omega(G)$.

Chromatic numbers and subgraphs

Lemma C: If H is a subgraph of G, then $\chi(H) \leq \chi(G)$.

Proof. If $\chi(G) = k$, then there is a proper coloring of G using k colors.
Let the vertices of H inherit their coloring from G.
This gives a proper coloring of H using k colors.
In turn, this implies $\chi(H) \leq k$.

If G contains a **clique** of size k (subgraph isomorphic to K_k), then what can we say about $\chi(G)$?

Definition. The **clique number** $\omega(G)$ is the size of the largest complete graph contained in G.

Theorem. For any graph G, $\chi(G) \geq \omega(G)$.

Proof. Apply Lemma C to the subgraph of G isomorphic to $K_{\omega(G)}$.

Chromatic numbers and subgraphs

Lemma C: If H is a subgraph of G, then $\chi(H) \leq \chi(G)$.

Proof. If $\chi(G) = k$, then there is a proper coloring of G using k colors.
Let the vertices of H inherit their coloring from G.
This gives a proper coloring of H using k colors.
In turn, this implies $\chi(H) \leq k$.

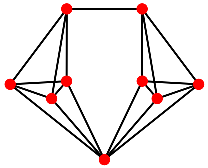
If G contains a **clique** of size k (subgraph isomorphic to K_k), then what can we say about $\chi(G)$?

Definition. The **clique number** $\omega(G)$ is the size of the largest complete graph contained in G.

Theorem. For any graph G, $\chi(G) \geq \omega(G)$.

Proof. Apply Lemma C to the subgraph of G isomorphic to $K_{\omega(G)}$.

Example. Calculate $\chi(G)$ for this graph G:



Critical graphs

How to prove $\chi(G) \geq k$?

Critical graphs

How to prove $\chi(G) \geq k$?

One way: Find a (small) subgraph H of G that requires k colors.

Critical graphs

How to prove $\chi(G) \geq k$?

One way: Find a (small) subgraph H of G that requires k colors.

Definition. A graph H is called **critical** if for every proper subgraph $J \subsetneq H$, then $\chi(J) < \chi(H)$.

Critical graphs

How to prove $\chi(G) \geq k$?

One way: Find a (small) subgraph H of G that requires k colors.

Definition. A graph H is called **critical** if for every proper subgraph $J \subsetneq H$, then $\chi(J) < \chi(H)$.

Theorem 2.1.2: Every graph G contains a critical subgraph H such that $\chi(H) = \chi(G)$.

Critical graphs

How to prove $\chi(G) \geq k$?

One way: Find a (small) subgraph H of G that requires k colors.

Definition. A graph H is called **critical** if for every proper subgraph $J \subsetneq H$, then $\chi(J) < \chi(H)$.

Theorem 2.1.2: Every graph G contains a critical subgraph H such that $\chi(H) = \chi(G)$.

(Stupid) Proof. If G is critical, stop. Define $H = G$.

Critical graphs

How to prove $\chi(G) \geq k$?

One way: Find a (small) subgraph H of G that requires k colors.

Definition. A graph H is called **critical** if for every proper subgraph $J \subsetneq H$, then $\chi(J) < \chi(H)$.

Theorem 2.1.2: Every graph G contains a critical subgraph H such that $\chi(H) = \chi(G)$.

(Stupid) Proof. If G is critical, stop. Define $H = G$.

If not, then there exists a proper subgraph G_1 of G with __________.

If G_1 is critical, stop. Define $H = G_1$.

Critical graphs

How to prove $\chi(G) \geq k$?

One way: Find a (small) subgraph H of G that requires k colors.

Definition. A graph H is called **critical** if for every proper subgraph $J \subsetneq H$, then $\chi(J) < \chi(H)$.

Theorem 2.1.2: Every graph G contains a critical subgraph H such that $\chi(H) = \chi(G)$.

(Stupid) Proof. If G is critical, stop. Define $H = G$.

If not, then there exists a proper subgraph G_1 of G with __________.

If G_1 is critical, stop. Define $H = G_1$.

If not, then there exists a proper subgraph G_2 of G_1 with ________.

If G_2 is critical, stop. Define $H = G_2$.

Critical graphs

How to prove $\chi(G) \geq k$?

One way: Find a (small) subgraph H of G that requires k colors.

Definition. A graph H is called **critical** if for every proper subgraph $J \subsetneq H$, then $\chi(J) < \chi(H)$.

Theorem 2.1.2: Every graph G contains a critical subgraph H such that $\chi(H) = \chi(G)$.

(Stupid) Proof. If G is critical, stop. Define $H = G$.

If not, then there exists a proper subgraph G_1 of G with ________.

If G_1 is critical, stop. Define $H = G_1$.

If not, then there exists a proper subgraph G_2 of G_1 with ______.

If G_2 is critical, stop. Define $H = G_2$.

If not, then there exists $\cdots$

Since G is finite, there will be some proper subgraph G_l of G_{l-1} such that G_l is critical and $\chi(G_l) = \chi(G_{l-1}) = \cdots = \chi(G)$.

Critical graphs

What do we know about critical graphs?

Critical graphs

What do we know about critical graphs?

Thm 2.1.1: Every critical graph is connected.

Critical graphs

What do we know about critical graphs?

Thm 2.1.1: Every critical graph is connected.

Thm 2.1.3: If G is critical and $\chi(G) = 4$, then $\deg(v) \geq 3$ for all v.

Critical graphs

What do we know about critical graphs?

Thm 2.1.1: Every critical graph is connected.

Thm 2.1.3: If G is critical and $\chi(G) = 4$, then $\deg(v) \geq 3$ for all v.

Proof. Suppose not. Then there is some $v \in V(G)$ with $\deg(v) \leq 2$. Remove v from G to create H.

Critical graphs

What do we know about critical graphs?

Thm 2.1.1: Every critical graph is connected.

Thm 2.1.3: If G is critical and $\chi(G) = 4$, then $\deg(v) \geq 3$ for all v.

Proof. Suppose not. Then there is some $v \in V(G)$ with $\deg(v) \leq 2$. Remove v from G to create H.

Similarly: If G is critical, then for all $v \in V(G)$, $\deg(v) \geq \chi(G) - 1$.

Bipartite graphs

Question. What is $\chi(C_n)$ when n is odd?

Answer.

Bipartite graphs

Question. What is $\chi(C_n)$ when n is odd?

Answer.

Definition. A graph is called **bipartite** if $\chi(G) \leq 2$.

Example. $K_{m,n}$, $\square_n$, Trees

Bipartite graphs

Question. What is $\chi(C_n)$ when n is odd?

Answer.

Definition. A graph is called **bipartite** if $\chi(G) \leq 2$.

Example. $K_{m,n}$, $\square_n$, Trees

Thm 2.1.6: G is bipartite $\iff$ every cycle in G has even length.

Bipartite graphs

Question. What is $\chi(C_n)$ when n is odd?

Answer.

Definition. A graph is called **bipartite** if $\chi(G) \leq 2$.

Example. $K_{m,n}$, $\square_n$, Trees

Thm 2.1.6: G is bipartite $\iff$ every cycle in G has even length.

$(\Rightarrow)$ Let G be bipartite. Assume that there is some cycle C of odd length contained in $G \ldots$

Proof of Theorem 2.1.6

($\Leftarrow$) Suppose that every cycle in G has even length. We want to show that G is bipartite. Consider the case when G is connected.

Proof of Theorem 2.1.6

($\Leftarrow$) Suppose that every cycle in G has even length. We want to show that G is bipartite. Consider the case when G is connected.

Plan: Construct a coloring on G and prove that it is proper.

Proof of Theorem 2.1.6

($\Leftarrow$) Suppose that every cycle in G has even length. We want to show that G is bipartite. Consider the case when G is connected.

Plan: Construct a coloring on G and prove that it is proper.

Choose some starting vertex x and color it blue. For every other vertex y, calculate the distance from y to x and then color y:

$$\begin{cases} \text{blue} & \text{if } d(x,y) \text{ is even.} \\ \text{red} & \text{if } d(x,y) \text{ is odd.} \end{cases}$$

Proof of Theorem 2.1.6

($\Leftarrow$) Suppose that every cycle in G has even length. We want to show that G is bipartite. Consider the case when G is connected.

Plan: Construct a coloring on G and prove that it is proper.

Choose some starting vertex x and color it blue. For every other vertex y, calculate the distance from y to x and then color y:

$$\begin{cases} \text{blue} & \text{if } d(x,y) \text{ is even.} \\ \text{red} & \text{if } d(x,y) \text{ is odd.} \end{cases}$$

Question: Is this a proper coloring of G?

If not, then

Proof of Theorem 2.1.6

($\Leftarrow$) Suppose that every cycle in G has even length. We want to show that G is bipartite. Consider the case when G is connected.

Plan: Construct a coloring on G and prove that it is proper.

Choose some starting vertex x and color it blue. For every other vertex y, calculate the distance from y to x and then color y:

$$\begin{cases} \text{blue} & \text{if } d(x,y) \text{ is even.} \\ \text{red} & \text{if } d(x,y) \text{ is odd.} \end{cases}$$

Question: Is this a proper coloring of G?

If not, then there are two adjacent vertices v and w of the same color.

Proof of Theorem 2.1.6

($\Leftarrow$) Suppose that every cycle in G has even length. We want to show that G is bipartite. Consider the case when G is connected.

Plan: Construct a coloring on G and prove that it is proper.

Choose some starting vertex x and color it blue. For every other vertex y, calculate the distance from y to x and then color y:

$$\begin{cases} \text{blue} & \text{if } d(x,y) \text{ is even.} \\ \text{red} & \text{if } d(x,y) \text{ is odd.} \end{cases}$$

Question: Is this a proper coloring of G?

If not, then there are two adjacent vertices v and w of the same color.

Claim 1: Their distance to the x is the same.

Proof of Theorem 2.1.6

($\Leftarrow$) Suppose that every cycle in G has even length. We want to show that G is bipartite. Consider the case when G is connected.

Plan: Construct a coloring on G and prove that it is proper.

Choose some starting vertex x and color it blue. For every other vertex y, calculate the distance from y to x and then color y:

$$\begin{cases} \text{blue} & \text{if } d(x, y) \text{ is even.} \\ \text{red} & \text{if } d(x, y) \text{ is odd.} \end{cases}$$

Question: Is this a proper coloring of G?

If not, then there are two adjacent vertices v and w of the same color.

Claim 1: Their distance to the x is the same.

Claim 2: There exists an odd cycle in G.

Proof of Theorem 2.1.6

($\Leftarrow$) Suppose that every cycle in G has even length. We want to show that G is bipartite. Consider the case when G is connected.

Plan: Construct a coloring on G and prove that it is proper.

Choose some starting vertex x and color it blue. For every other vertex y, calculate the distance from y to x and then color y:

$$\begin{cases} \text{blue} & \text{if } d(x,y) \text{ is even.} \\ \text{red} & \text{if } d(x,y) \text{ is odd.} \end{cases}$$

Question: Is this a proper coloring of G?

If not, then there are two adjacent vertices v and w of the same color.

Claim 1: Their distance to the x is the same.

Claim 2: There exists an odd cycle in G.

This contradicts our hypothesis, so a 2-coloring exists; G is bipartite.

Edge Coloring

Parallel to the idea of vertex coloring is the idea of edge coloring.

Edge Coloring

Parallel to the idea of vertex coloring is the idea of edge coloring.

Definition. An **edge coloring** of a graph G is a labeling of the edges of G with colors. [Technically, it is a function $f : E(G) \to \{1, 2, \ldots, l\}$.]

Edge Coloring

Parallel to the idea of vertex coloring is the idea of edge coloring.

Definition. An **edge coloring** of a graph G is a labeling of the edges of G with colors. [Technically, it is a function $f : E(G) \to \{1, 2, \ldots, l\}$.]

Definition. A **proper** edge coloring of G is an edge coloring of G such that no two *adjacent edges* are colored the same.

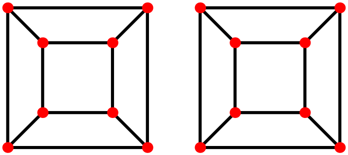
Edge Coloring

Parallel to the idea of vertex coloring is the idea of edge coloring.

Definition. An **edge coloring** of a graph G is a labeling of the edges of G with colors. [Technically, it is a function $f : E(G) \to \{1, 2, \ldots, l\}$.]

Definition. A **proper** edge coloring of G is an edge coloring of G such that no two *adjacent edges* are colored the same.

Example. Cube graph ($\square_3$):



We can properly edge color $\square_3$ with ____ colors and no fewer.

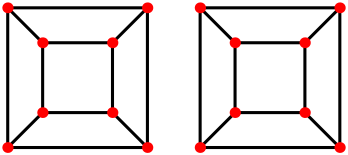
Edge Coloring

Parallel to the idea of vertex coloring is the idea of edge coloring.

Definition. An **edge coloring** of a graph G is a labeling of the edges of G with colors. [Technically, it is a function $f : E(G) \to \{1, 2, \ldots, l\}$.]

Definition. A **proper** edge coloring of G is an edge coloring of G such that no two *adjacent edges* are colored the same.

Example. Cube graph ($\square_3$):



We can properly edge color $\square_3$ with ____ colors and no fewer.

Definition. The minimum number of colors necessary to properly edge color a graph G is called the **edge chromatic number** of G, denoted $\chi'(G) =$ "chi prime".

Edge coloring theorems

Thm 2.2.1: For any graph G, $\chi'(G) \geq \Delta(G)$.

Edge coloring theorems

Thm 2.2.1: For any graph G, $\chi'(G) \geq \Delta(G)$.

Thm 2.2.2: Vizing's Theorem:

For any graph G, $\chi'(G)$ equals either $\Delta(G)$ or $\Delta(G) + 1$.

Edge coloring theorems

Thm 2.2.1: For any graph G, $\chi'(G) \geq \Delta(G)$.

Thm 2.2.2: Vizing's Theorem:

For any graph G, $\chi'(G)$ equals either $\Delta(G)$ or $\Delta(G) + 1$.

Proof. Hard. (See reference [24] if interested.)

Edge coloring theorems

Thm 2.2.1: For any graph G, $\chi'(G) \geq \Delta(G)$.

Thm 2.2.2: Vizing's Theorem:

For any graph G, $\chi'(G)$ equals either $\Delta(G)$ or $\Delta(G) + 1$.

Proof. Hard. (See reference [24] if interested.)

Consequence: To determine $\chi'(G)$,

Edge coloring theorems

Thm 2.2.1: For any graph G, $\chi'(G) \geq \Delta(G)$.

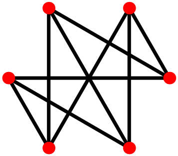
Thm 2.2.2: Vizing's Theorem:

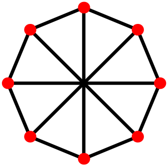
For any graph G, $\chi'(G)$ equals either $\Delta(G)$ or $\Delta(G) + 1$.

Proof. Hard. (See reference [24] if interested.)

Consequence: To determine $\chi'(G)$,

Fact: **Most** 3-regular graphs have edge chromatic number 3.





Snarks

Definition. Another name for 3-regular is **cubic**.

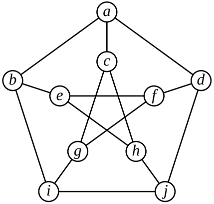
Definition. A **snark** is a *bridgeless* cubic graph with edge chromatic number 4.

Snarks

Definition. Another name for 3-regular is **cubic**.

Definition. A **snark** is a *bridgeless* cubic graph with edge chromatic number 4.

Example. The Petersen graph P is a snark.

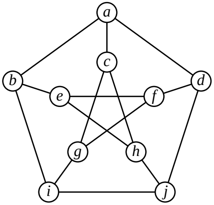


Snarks

Definition. Another name for 3-regular is **cubic**.

Definition. A **snark** is a *bridgeless* cubic graph with edge chromatic number 4.

Example. The Petersen graph P is a snark. It is 3-regular. ✓

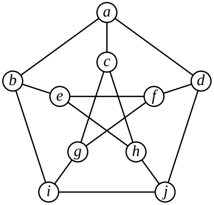


Snarks

Definition. Another name for 3-regular is **cubic**.

Definition. A **snark** is a *bridgeless* cubic graph with edge chromatic number 4.

Example. The Petersen graph P is a snark. It is 3-regular. ✓ Let us prove that it can not be colored with three colors.

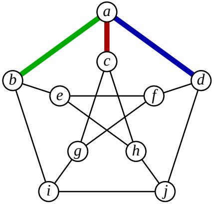


Snarks

Definition. Another name for 3-regular is **cubic**.

Definition. A **snark** is a *bridgeless* cubic graph with edge chromatic number 4.

Example. The Petersen graph P is a snark. It is 3-regular. ✓
Let us prove that it can not be colored with three colors.
Assume you can color it with three colors. WLOG, assume ab, ac, ad.

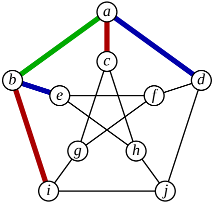


Snarks

Definition. Another name for 3-regular is **cubic**.

Definition. A **snark** is a *bridgeless* cubic graph with edge chromatic number 4.

Example. The Petersen graph P is a snark. It is 3-regular. ✓
Let us prove that it can not be colored with three colors.
Assume you can color it with three colors. WLOG, assume ab, ac, ad.
Either **Case 1**: be and bi

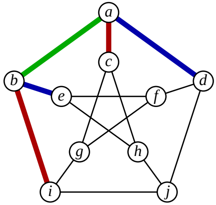


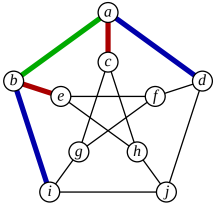
Snarks

Definition. Another name for 3-regular is **cubic**.

Definition. A **snark** is a *bridgeless* cubic graph with edge chromatic number 4.

Example. The Petersen graph P is a snark. It is 3-regular. ✓
Let us prove that it can not be colored with three colors.
Assume you can color it with three colors. WLOG, assume ab, ac, ad.
Either **Case 1**: be and bi or **Case 2**: be and bi.



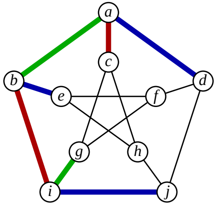


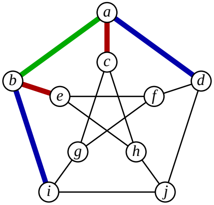
Snarks

Definition. Another name for 3-regular is **cubic**.

Definition. A **snark** is a *bridgeless* cubic graph with edge chromatic number 4.

Example. The Petersen graph P is a snark. It is 3-regular. ✓
Let us prove that it can not be colored with three colors.
Assume you can color it with three colors. WLOG, assume *ab*, *ac*, *ad*.
Either **Case 1**: *be* and *bi* or **Case 2**: *be* and *bi*.
Either **Case 1a**: *ig* and *ij*



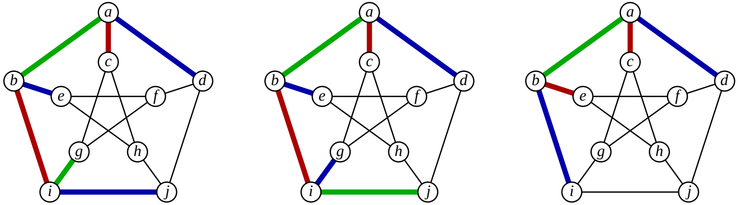


Snarks

Definition. Another name for 3-regular is **cubic**.

Definition. A **snark** is a *bridgeless* cubic graph with edge chromatic number 4.

Example. The Petersen graph P is a snark. It is 3-regular. ✓
Let us prove that it can not be colored with three colors.
Assume you can color it with three colors. WLOG, assume ab, ac, ad.
Either **Case 1**: be and bi or **Case 2**: be and bi.
Either **Case 1a**: ig and ij or **Case 1b**: ig and ij.

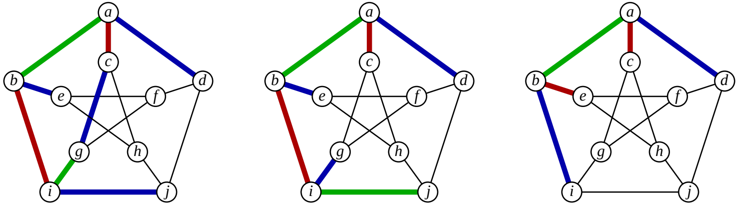


Snarks

Definition. Another name for 3-regular is **cubic**.

Definition. A **snark** is a *bridgeless* cubic graph with edge chromatic number 4.

Example. The Petersen graph P is a snark. It is 3-regular. ✓
Let us prove that it can not be colored with three colors.
Assume you can color it with three colors. WLOG, assume ab, ac, ad.
Either **Case 1**: be and bi or **Case 2**: be and bi.
Either **Case 1a**: ig and ij or **Case 1b**: ig and ij.

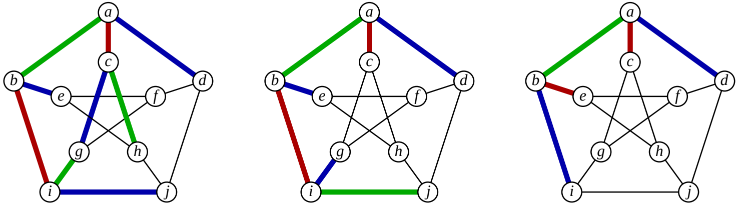


Snarks

Definition. Another name for 3-regular is **cubic**.

Definition. A **snark** is a *bridgeless* cubic graph with edge chromatic number 4.

Example. The Petersen graph P is a snark. It is 3-regular. ✓
Let us prove that it can not be colored with three colors.
Assume you can color it with three colors. WLOG, assume ab, ac, ad.
Either **Case 1**: be and bi or **Case 2**: be and bi.
Either **Case 1a**: ig and ij or **Case 1b**: ig and ij.

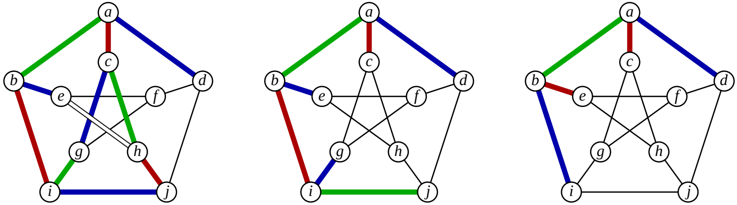


Snarks

Definition. Another name for 3-regular is **cubic**.

Definition. A **snark** is a *bridgeless* cubic graph with edge chromatic number 4.

Example. The Petersen graph P is a snark. It is 3-regular. ✓
Let us prove that it can not be colored with three colors.
Assume you can color it with three colors. WLOG, assume ab, ac, ad.
Either **Case 1**: be and bi or **Case 2**: be and bi.
Either **Case 1a**: ig and ij or **Case 1b**: ig and ij.

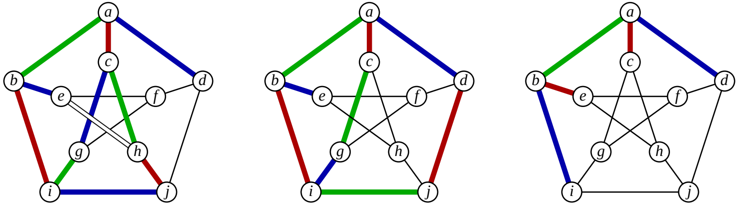


Snarks

Definition. Another name for 3-regular is **cubic**.

Definition. A **snark** is a *bridgeless* cubic graph with edge chromatic number 4.

Example. The Petersen graph P is a snark. It is 3-regular. ✓
Let us prove that it can not be colored with three colors.
Assume you can color it with three colors. WLOG, assume ab, ac, ad.
Either **Case 1**: be and bi or **Case 2**: be and bi.
Either **Case 1a**: ig and ij or **Case 1b**: ig and ij.

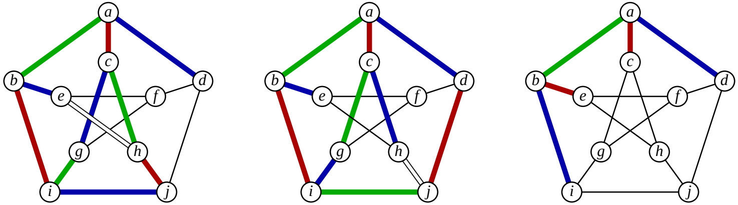


Snarks

Definition. Another name for 3-regular is **cubic**.

Definition. A **snark** is a *bridgeless* cubic graph with edge chromatic number 4.

Example. The Petersen graph P is a snark. It is 3-regular. ✓
Let us prove that it can not be colored with three colors.
Assume you can color it with three colors. WLOG, assume ab, ac, ad.
Either **Case 1**: be and bi or **Case 2**: be and bi.
Either **Case 1a**: ig and ij or **Case 1b**: ig and ij.

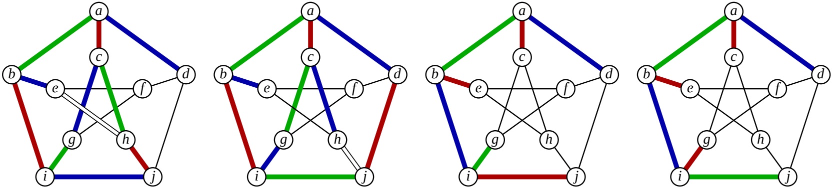


Snarks

Definition. Another name for 3-regular is **cubic**.

Definition. A **snark** is a *bridgeless* cubic graph with edge chromatic number 4.

Example. The Petersen graph P is a snark. It is 3-regular. ✓
Let us prove that it can not be colored with three colors.
Assume you can color it with three colors. WLOG, assume ab, ac, ad.
Either **Case 1**: be and bi or **Case 2**: be and bi.
Either **Case 1a**: ig and ij or **Case 1b**: ig and ij. **Cases 2a, 2b**

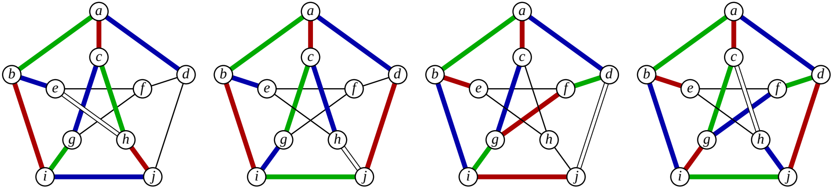


Snarks

Definition. Another name for 3-regular is **cubic**.

Definition. A **snark** is a *bridgeless* cubic graph with edge chromatic number 4.

Example. The Petersen graph P is a snark. It is 3-regular. ✓
Let us prove that it can not be colored with three colors.
Assume you can color it with three colors. WLOG, assume ab, ac, ad.
Either **Case 1**: be and bi or **Case 2**: be and bi.
Either **Case 1a**: ig and ij or **Case 1b**: ig and ij. **Cases 2a, 2b**

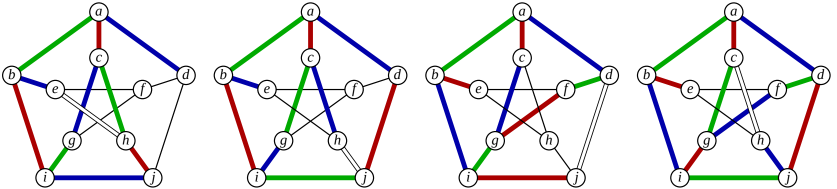


Snarks

Definition. Another name for 3-regular is **cubic**.

Definition. A **snark** is a *bridgeless* cubic graph with edge chromatic number 4.

Example. The Petersen graph P is a snark. It is 3-regular. ✓
Let us prove that it can not be colored with three colors.
Assume you can color it with three colors. WLOG, assume *ab*, *ac*, *ad*.
Either **Case 1**: *be* and *bi* or **Case 2**: *be* and *bi*.
Either **Case 1a**: *ig* and *ij* or **Case 1b**: *ig* and *ij*. **Cases 2a, 2b**



In all cases, it is not possible to edge color with 3 colors, so $\chi'(G) = 4$.

The edge chromatic number of complete graphs

Goal: Determine $\chi'(K_n)$ for all n.

The edge chromatic number of complete graphs

Goal: Determine $\chi'(K_n)$ for all n.

Vertex Degree Analysis: The degree of every vertex in K_n is ____.

The edge chromatic number of complete graphs

Goal: Determine $\chi'(K_n)$ for all n.

Vertex Degree Analysis: The degree of every vertex in K_n is ____.

Vizing's theorem implies that $\chi'(K_n) =$ ____ or ____.

If $\chi'(K_n) =$ ____, then each vertex has an edge leaving of each color.

The edge chromatic number of complete graphs

Goal: Determine $\chi'(K_n)$ for all n.

Vertex Degree Analysis: The degree of every vertex in K_n is ____.

Vizing's theorem implies that $\chi'(K_n) =$ ____ or ____.

If $\chi'(K_n) =$ ____, then each vertex has an edge leaving of each color.

Question. How many red edges are there?

The edge chromatic number of complete graphs

Goal: Determine $\chi'(K_n)$ for all n.

Vertex Degree Analysis: The degree of every vertex in K_n is ____.

Vizing's theorem implies that $\chi'(K_n) =$____ or ____.

If $\chi'(K_n) =$____, then each vertex has an edge leaving of each color.

Question. How many red edges are there?

This is only an integer when:

So, the best we can expect is that $\begin{cases} \chi'(K_{2n}) = \\ \chi'(K_{2n-1}) = \end{cases}$

The edge chromatic number of complete graphs

Thm 2.2.3: $\chi'(K_{2n}) = 2n - 1$.

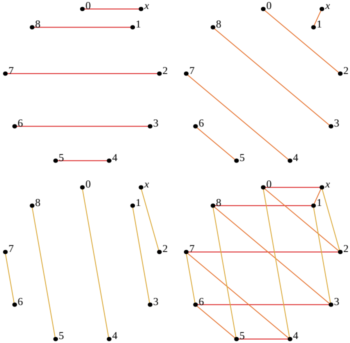
Proof. We prove this using the *turning trick*.

The edge chromatic number of complete graphs

Thm 2.2.3: $\chi'(K_{2n}) = 2n - 1$.

Proof. We prove this using the *turning trick*.

Label the vertices of K_{2n}
$0, 1, \ldots, 2n-2, x$. Now,
Connect 0 with x,
Connect 1 with $2n-2$,
⋮
Connect $n-1$ with n.



The edge chromatic number of complete graphs

Thm 2.2.3: $\chi'(K_{2n}) = 2n - 1$.

Proof. We prove this using the *turning trick*.

Label the vertices of K_{2n}

$0, 1, \ldots, 2n-2, x$. Now,

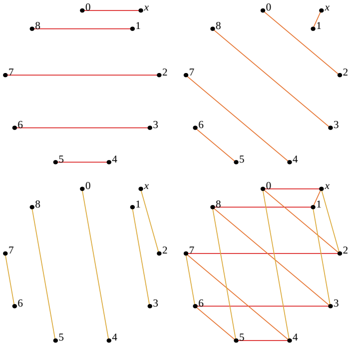
Connect 0 with x,

Connect 1 with $2n-2$,

⋮

Connect $n-1$ with n.

Now **turn** the inside edges.



The edge chromatic number of complete graphs

Thm 2.2.3: $\chi'(K_{2n}) = 2n - 1$.

Proof. We prove this using the *turning trick*.

Label the vertices of K_{2n}

$0, 1, \ldots, 2n-2, x$. Now,

Connect 0 with x,

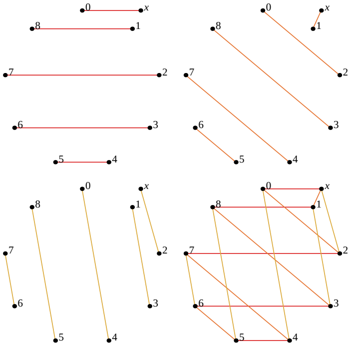
Connect 1 with $2n-2$,

⋮

Connect $n-1$ with n.

Now **turn** the inside edges.

And do it again. (and again, ...)

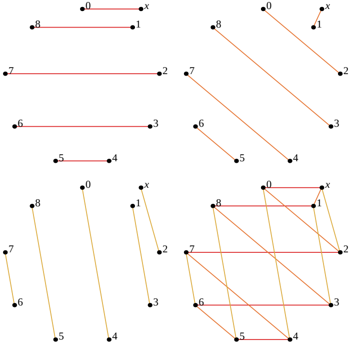


The edge chromatic number of complete graphs

Thm 2.2.3: $\chi'(K_{2n}) = 2n - 1$.

Proof. We prove this using the *turning trick*.

Label the vertices of K_{2n}
$0, 1, \ldots, 2n-2, x$. Now,
Connect 0 with x,
Connect 1 with $2n-2$,
⋮
Connect $n-1$ with n.



Now **turn** the inside edges.
And do it again. (and again, ...)

Each time, new edges are used.
This is because each of the edges is a different "circular length": vertices are at circ. distance 1, 3, 5, ..., 4, 2 from each other, and x is connected to a different vertex each time.

The edge chromatic number of complete graphs

Theorem 2.2.4: $\chi'(K_{2n-1}) = 2n - 1.$

The edge chromatic number of complete graphs

Theorem 2.2.4: $\chi'(K_{2n-1}) = 2n - 1$.

This construction also gives a way to edge color K_{2n-1} with $2n - 1$ colors—simply delete vertex x!

The edge chromatic number of complete graphs

Theorem 2.2.4: $\chi'(K_{2n-1}) = 2n - 1$.

This construction also gives a way to edge color K_{2n-1} with $2n - 1$ colors—simply delete vertex x!

This is related to the area of combinatorial designs.

Question. Is it possible for six tennis players to play one match per day in a five-day tournament in such a way that each player plays each other player once?

The edge chromatic number of complete graphs

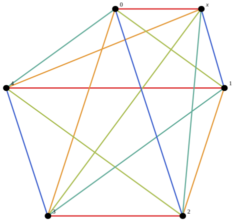
Theorem 2.2.4: $\chi'(K_{2n-1}) = 2n - 1$.

This construction also gives a way to edge color K_{2n-1} with $2n-1$ colors—simply delete vertex x!

This is related to the area of combinatorial designs.

Question. Is it possible for six tennis players to play one match per day in a five-day tournament in such a way that each player plays each other player once?

Day 1	0x	14	23
Day 2	1x	20	34
Day 3	2x	31	40
Day 4	3x	42	01
Day 5	4x	03	12



Theorem 2.2.3 proves there is such a tournament for all even numbers.