Special subgraphs

Definition. (pp. 20,34) Let G be a graph.
A subgraph H is **spanning** if H contains all of G's vertices.

- If H is spanning and also a tree, H is called a **spanning tree**.
- If H is spanning and also r-regular, then H is called an r-**factor**.
  A 1-factor is also called a **perfect matching**.
  For a graph G to have a perfect matching, ________________.

Definition. A **decomposition** of a graph G is a set of subgraphs $H_1, \ldots, H_k$ that partition of the edges of G. That is, for all i and j,

$$\bigcup_{1 \leq i \leq k} H_i = G \text{ and } E(H_i) \cap E(H_j) = \emptyset.$$

Definition. An H-**decomposition** is a decomposition of G such that each subgraph H_i in the decomposition is isomorphic to H.

Perfect Matching Decomposition

Definition. A **perfect matching decomposition** is a decomposition such that each subgraph H_i in the decomposition is a perfect matching.

Theorem: For a k-regular graph G,
G has a perfect matching decomposition if and only if $\chi'(G) = k$.

Proof:

There exists a decomposition of G into a set of k perfect matchings.

$$\Updownarrow$$

There exists a coloring of the edges of G where each vertex is incident to edges of each of k different colors.

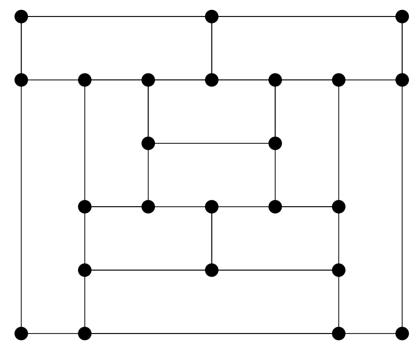
$$\Updownarrow$$

$$\chi'(G) = k$$

Corollary: K_{2n+1} has a perfect matching decomposition.
Corollary: A snark has no perfect matching decomposition.

Hamiltonian Cycles

Definition. A **Hamiltonian cycle** C in a graph G is a cycle containing every vertex of G.



Definition. A **Hamiltonian path** P in a graph G is a path containing every vertex of G.

Theorem: If G has a Ham'n cycle, then G has a Ham'n path.
Proof:

An arbitrary graph may or may not contain a Hamiltonian cycle/path. This is very hard to determine in general!

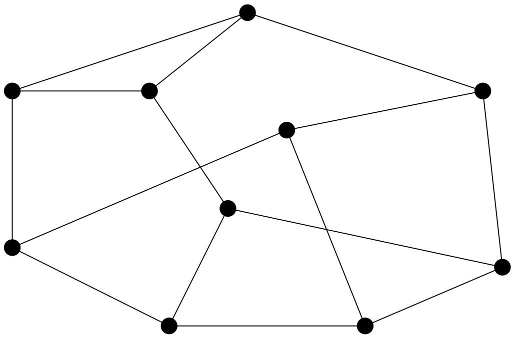
⋆ Important: Paths and cycles do not use any vertex or edge twice. ⋆

Hamiltonian Cycles

Theorem 2.3.5: A snark has no Hamiltonian cycle.

Fact: A snark has an even number of vertices.

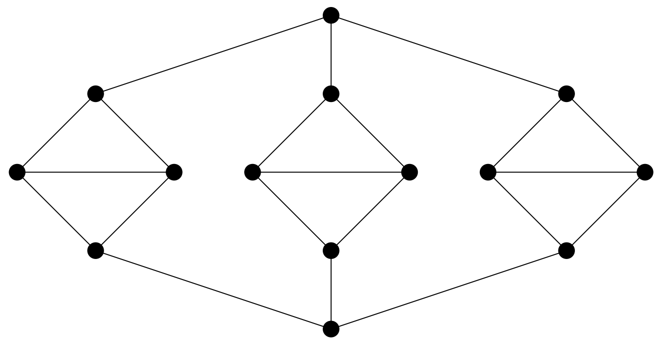
Proof: Suppose that a graph G is a snark and contains a Hamiltonian cycle.
That is, G contains C, visiting each vertex once.
Remove the edges of C; what remains?



Consider the coloring of G where the remaining edges are colored yellow and the edges in the cycle are colored alternating between blue and red. This is a proper 3-edge-coloring of G, a contradiction.

The converse is not true!

Example: Book Figure 2.3.4.



Cycle Decompositions

Definition. A **cycle decomposition** is a decomposition such that each subgraph H_i in the decomposition is a cycle.

Theorem: Let G be a graph that has a cycle decomposition. Then every vertex of G has even degree.

Proof: In the cycle decomposition, each cycle H_i contributes two to the degree of each of its vertices. For a vertex v in G, its degree is the sum of its degree over all subgraphs H_i, which must be even.

Definition. A **Hamiltonian cycle decomposition** is a decomposition such that each subgraph H_i is a Hamiltonian cycle.

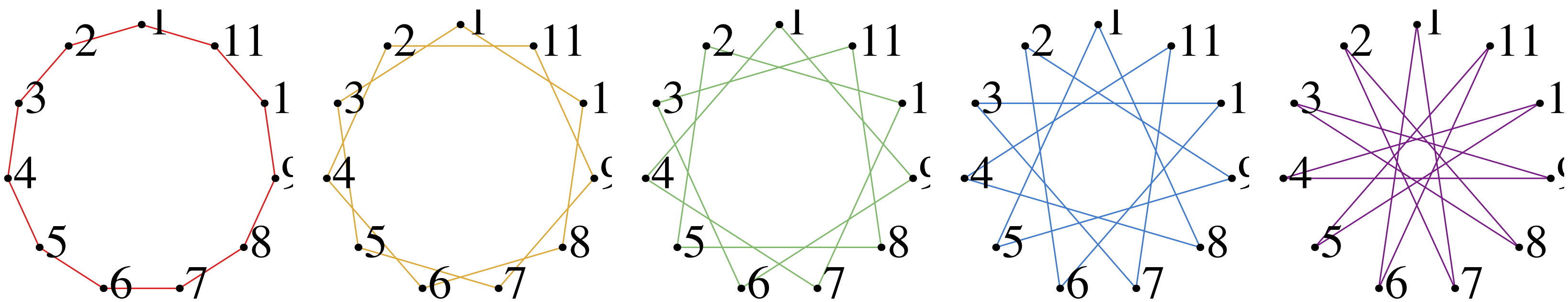
Question: Which graphs have a Hamiltonian cycle decomposition?

Which complete graphs?

Hamiltonian Cycle Decomposition

Example: K_7 has a Hamiltonian cycle decomposition.

Example: K_{11} has a Hamiltonian cycle decomposition.

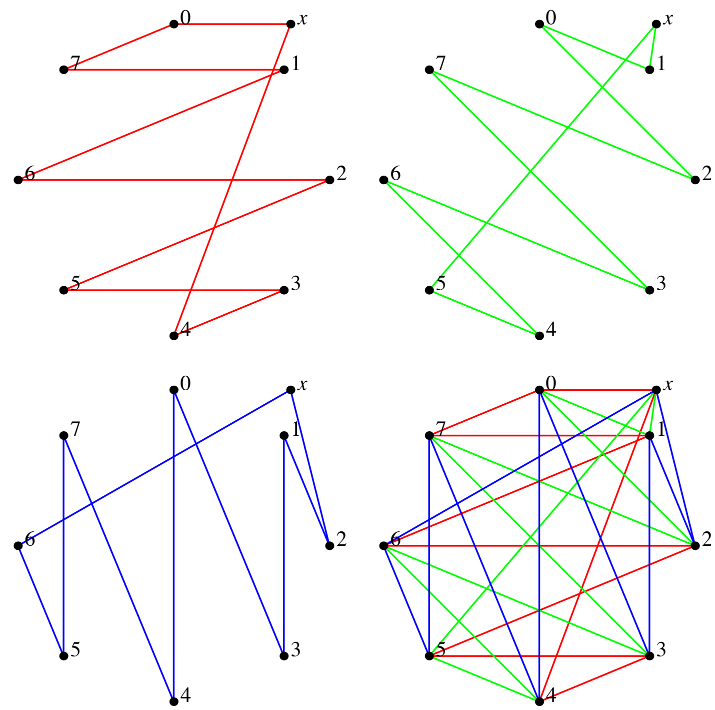


However: This construction does not work with K_9.

Hamiltonian Cycle Decomposition

Theorem 2.3.1: K_{2n+1} has a Hamiltonian cycle decomposition.

Proof: This proof uses another instance of a "turning trick". Place vertices 0 through $2n$ in a circle and draw a zigzag path visiting all the vertices in the circle. Connect the ends of the path to vertex x to form a Ham. cycle. As you rotate the zigzag path n times, you visit each edge of K_{2n+1} once to form a Ham'n cycle decomposition.



As a corollary:

Theorem 2.3.3: K_{2n} has a Hamiltonian path decomposition.