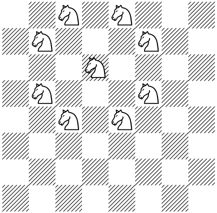
Knight's Tours

In chess, a knight (♘) is a piece that moves in an "L": two spaces over and one space to the side.



Question. Is it possible for a knight to start on some square and, by a series of valid knight moves, visit each square on an 8×8 chessboard once? (How about return to where it started?)

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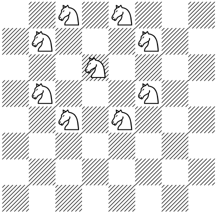

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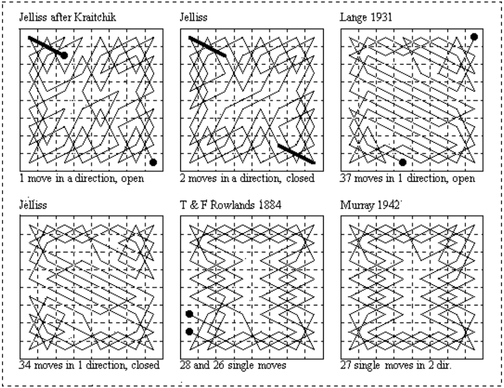
Definition. A path of the first kind is called an **open knight's tour**. A cycle of the second kind is called a **closed knight's tour**.

8 × 8 Knight's Tour



Source: `http://www.mayhematics.com/t/8g.htm`

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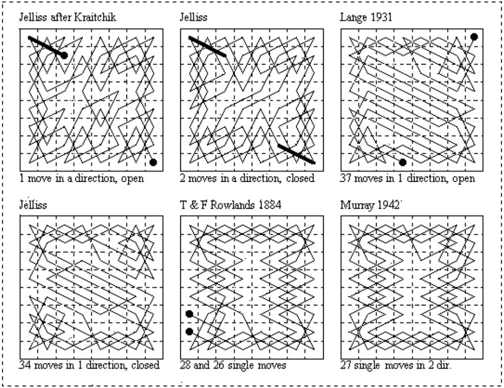

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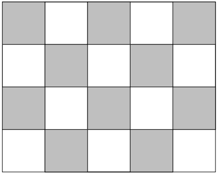
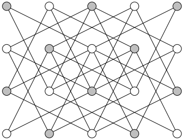
Question. Are there any knight's tours on an $m \times n$ chessboard?

The Graph Theory of Knight's Tours

For any board we can draw a corresponding knight move graph: Create a vertex for every square on the board and create edges between vertices that are a knight's move away.



An open/closed knight's tour on the board



A knight move always alternates between white and black squares. Therefore, a knight move graph is always ______________.

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Knight's Tour Theorem

Theorem. An $m \times n$ chessboard with $m \leq n$ has a *closed* knight's tour unless one or more of these conditions holds:

1. m and n are both odd.
2. $m = 1$, 2, or 4.
3. $m = 3$ and $n = 4$, 6, or 8.

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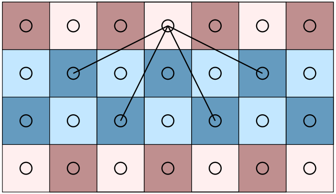
Case 1. When m and n are both odd,

Case 2. When $m = 1$ or 2, the knight move graph is not connected.

Knight's Tour Theorem

Case 2. When $m = 4$, draw the knight move graph G.



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Since G is bipartite, C alternates between white and black vertices.

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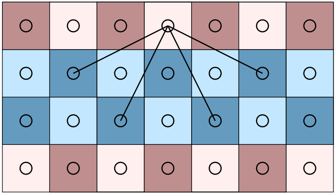

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Notice that every red vertex in C is adjacent to only blue vertices.
And, there are the same number of red and blue vertices.

So, C must alternate between red and blue vertices.

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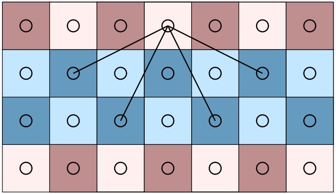

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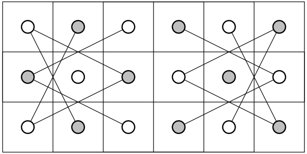
Notice that every red vertex in C is adjacent to only blue vertices.
And, there are the same number of red and blue vertices.

So, C must alternate between red and blue vertices. This means:
All vertices of C are "white and red" or "black and blue".

Knight's Tour Theorem

Case 3. 3×4 is covered by Case 2. Consider the 3×6 board:



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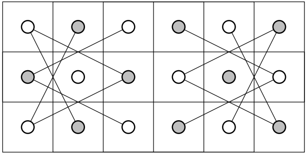

Assume that there is a Hamiltonian cycle C in G.
Then, C visits each vertex v and uses two of v's incident edges.
If $\deg(v) = 2$, then both of v's incident edges are in C.

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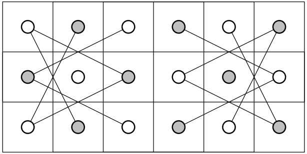

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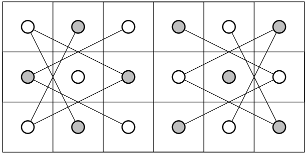

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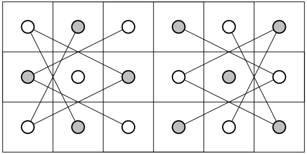

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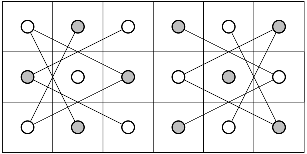

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The 3×8 case is similar, and part of your homework.

See also: "Knight's Tours on a Torus", by J. J. Watkins, R. L. Hoenigman