The Origins of Graph Theory

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We can model this situation with a graph:

Question. Can we draw this graph without lifting our pencil?

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Definition. The **length** of a "walk" is the number of *edges* involved.

Remark. In a simple graph, the smallest cycle possible is of length 3. In a pseudograph, there may exist cycles of length 1 and 2.

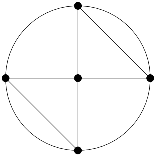
Definition. The **degree** of a vertex v is the number of edges incident with v; loops count twice!

Eulerian Circuits

Definition. An **Eulerian circuit** C in a graph G is a circuit containing every edge of G.
Definition. An **Eulerian trail** T in a graph G is a trail containing every edge of G.

A graph with an Eulerian circuit does not have an Eulerian trail.

⋆ Important: Trails and circuits do not use any edge twice. ⋆



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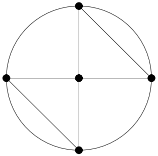

So the Königsberg bridge problem in the language of graph theory is:

Is there an Eulerian circuit in the corresponding pseudograph?

Characterization of Graphs with Eulerian Circuits

There is a simple way to determine if a graph has an Eulerian circuit.

Theorems 3.1.1 and 3.1.2. Let G be a pseudograph that is connected* *except possibly for isolated vertices*. Then,
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($\Leftarrow$) Hierholzer, 1873. This is harder; we need the following lemma.

Proof of Lemma 3.1.3

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The trail must eventually return to A, giving us a circuit.

Proof of Theorem 3.1.2

⋆ Each vertex in G has even degree $\Rightarrow$ G has an Eulerian circuit ⋆

Find the longest circuit C in G. If C uses every edge, we are done. If not, we'll show a contradiction to the maximality of C.

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Define a new circuit $C' = \cdots e_1 A f_2 \cdots f_1 A e_2 \cdots$.

C' is a longer circuit in G than C, contradicting C's maximality. $\square$

Other related theorems

Theorem 3.1.6. Let G be a connected* pseudograph. Then, G has an Eulerian trail $\Leftrightarrow$ G has exactly two vertices of odd degree.

Proof. Let x and y be the two vertices of odd degree. Add edge xy to G; $G + xy$ is a pseudograph with each vertex of even degree. By Theorem 3.1.2, there exists an Eulerian circuit in $G + xy$. Remove xy from the circuit and you have an Eulerian trail in G.

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Theorem 3.1.5. A pseudograph G has a decomposition into cycles if and only if every vertex has even degree.

Application: de Bruijn sequences

Consider the following example of a **de Bruijn sequence**:

$$0000110101111001$$

Each of the sixteen binary sequences of length 4 are present (where we allow cycling):

0000	0100	1000	1100
0001	0101	1001	1101
0010	0110	1010	1110
0011	0111	1011	1111

This is the most compact way to represent these sixteen sequences.

Sequence definitions

Definition. An **alphabet** is a set $\mathcal{A} = \{a_1, \ldots, a_k\}$.

Definition. A **sequence** or **word** from $\mathcal{A}$ is a succession
$S = s_1 s_2 s_3 \cdots s_l$, where each $s_i \in \mathcal{A}$; l is the **length** of S.

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Theorem. A de Bruijn sequence of any order n on any alphabet $\mathcal{A}$ always exists.

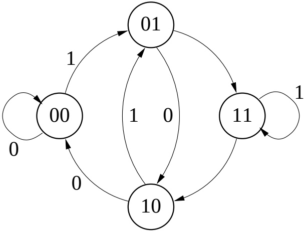
Proof. Use the theory of Eulerian circuits on certain graphs:

de Bruijn graphs

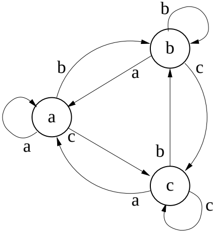
Definition. The **de Bruijn graph** of order n on $\mathcal{A} = \{a_1, a_2, \ldots, a_k\}$ is a directed pseudograph that has as its vertices words of $\mathcal{A}$ of length $n-1$.

Examples.

The binary de Bruijn graph of order 3



The de Bruijn graph of order 2 on the alphabet $\mathcal{A} = \{a, b, c\}$.



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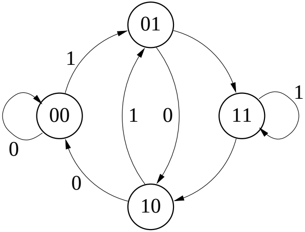

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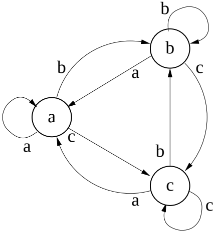

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$$\boxed{b_1 b_2 \cdots b_{n-1}} \xrightarrow{a_i} \boxed{b_2 \cdots b_{n-1} a_i}$$

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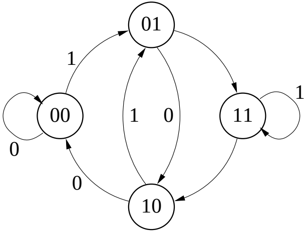

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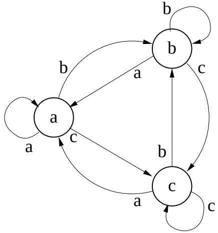

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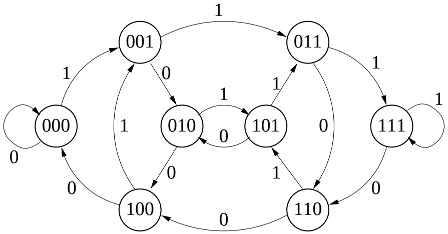
By construction, the sequence of the $n-1$ labels of edges visited before arriving at a vertex is exactly the name of the vertex. The word formed by this name followed by the label of an outgoing edge is a word of $\mathcal{A}$ of length n and is different for every edge of C. This implies that every sequence appears as a consecutive subseq. of S.

Example: The binary de Bruijn graph of order 4



1. Find an Eulerian circuit in this graph.
2. Write down the corresponding sequence.
3. Verify that it is a de Bruijn sequence. (use chart, p.65)
4. Convince yourself that the name of a vertex is the same as the sequence formed by the three previous edges.