Connectivity

When a graph is **disconnected**, it does have connected subgraphs.

Definition. A **connected component** H of G is a maximally connected subgraph.

More precisely,
For every subgraph K such that $H \subsetneq K \subset G$, K is not connected.

Interpretation: If we try to add any more vertices and/or edges of G into H, the result would be disconnected.

Vertex and edge connectivity

Definitions. Let G be a connected graph.

1. A **cut vertex** is a vertex v such that $G \setminus v$ is disconnected.
2. A **cut set** is a set of vertices X such that $G \setminus X$ is disconnected.
3. A **bridge** is an edge e such that $G \setminus e$ is disconnected.
4. A **disconnecting set** is a set of edges D such that $G \setminus D$ is disconnected.

- If you delete a bridge from a graph, __________________________.
- If you delete a cut vertex from a graph, __________________________.

Vertex and edge connectivity

Definition. The **connectivity** of G (denoted $\kappa(G) =$ "kappa") is the size of the minimum cut set in G.

We need the convention: $\kappa(K_n) = n - 1$ (incl. $\kappa(\{v\}) = 0$)

- $\kappa(G) = 0 \iff G$ is disconnected or G is a single vertex.
- $\kappa(G) \geq 2 \iff G$ has no cut vertex.

Definition. The **edge connectivity** of G (denoted $\kappa'(G)$) is the size of the minimum disconnecting set in G.

- $\kappa'(G) = 0 \iff G$ is disconnected or G is a single vertex.
- $\kappa'(G) \geq 2 \iff G$ has no bridge.

Fact: For all graphs G, $\kappa(G) \leq \kappa'(G) \leq \delta(G)$. ($\leftarrow$ min. vtx. degree)

Minimum vs. Minimal

We have just had two definitions related to the concept of "smallest". An important concept is the distinction between minim*um* and minim*al*.

Minimum refers to an element of absolute smallest size.
(of *ALL* elts with *property*, this is smallest.)

Minimal refers to an element of relative smallest size.
(for *THIS* elt with *property*, no subset has property.)

Example. minimum vs. minimal disconnecting set:

Example. maximum vs. maximal path in a graph:

Girth and diameter

We now discuss additional **graph statistics**, numerical properties that can be calculated for every graph and can be used to distinguish between non-isomorphic graphs.

Definition. The **girth** of a graph G, denoted $\mathrm{g}(G)$, is the length of the shortest cycle contained in G. If no cycles exist, $\mathrm{g}(G) = \infty$.

Definition. Let $x, y \in V(G)$. The **distance** from x to y, denoted $d(x, y)$, is the length of the shortest path in G from x to y.

If no path exists between x and y, then define $d(x, y) = \infty$.

Definition. The **diameter** of a graph G, denoted $\mathrm{diam}(G)$, is the *maximum* distance between any two vertices of G.

★ This definition is hard to apply! Calculate a max over a min. ★

Cliques and independent sets

Definitions

A **clique** is a set of vertices in G, *all* of which are adjacent.

An **independent set** is a set of vertices in G, *none* of which are adjacent.

A **vertex cover** is a set of vertices X in G such that X contains (at least) one endpoint of every edge in G.

$\star$ Note: This set of vertices covers the *edges* of G. $\star$

Definitions

The **clique number** of G, denoted $\omega(G)$ ('omega'), is the size of the largest clique in G.

The **independence number** of G, denoted $\alpha(G)$ ('alpha'), is the size of the largest independent set of G.

The **vertex cover number** of G, denoted $\beta(G)$ ('beta'), is the size of the smallest vertex cover of G.

Statistical relationships

Theorems: Let G be a graph and suppose $X \subset V(G)$.

1. X is a clique in $G \iff X$ is an independent set in G^c.
2. X is an independent set in $G \iff X^c$ is a vertex cover in G.
3. For all graphs G, $\alpha(G) + \beta(G) = |V(G)|$.
4. We can bound the chromatic number: $\chi(G) \geq \dfrac{|V(G)|}{\alpha(G)}$

Recap: Consider the following statistics to tell two graphs apart.

$\lvert V\rvert$, $\lvert E\rvert$, deg. seq.	$\Delta(G)$: max vtx deg	$\delta(G)$: min vtx deg
$\chi(G)$: chrom	$\kappa(G)$: vtx connect	$g(G)$: girth
$\chi'(G)$: edge chrom	$\kappa'(G)$: edge connect	diam(G): diameter
$\omega(G)$: clique #	$\alpha(G)$: indep #	$\beta(G)$: vtx cvr #