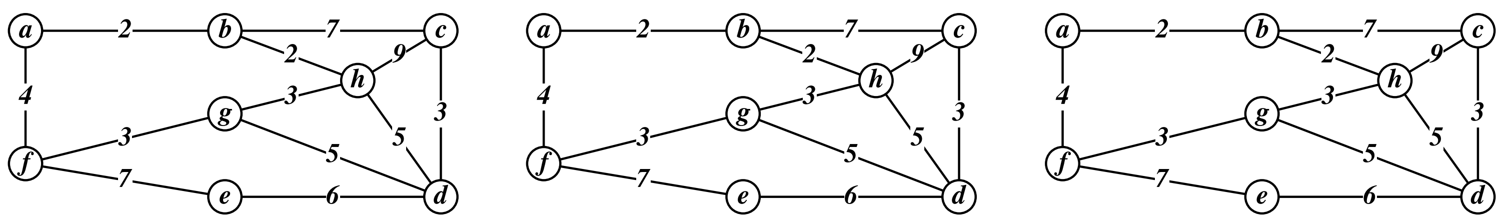
Minimum-weight spanning trees

Motivation: Create a connected network as cheaply as possible.

- Think: Setting up electrical grid or road network.
- Some connections are cheaper than others.
- Only need to minimally connect the vertices.

Definition. A **weighted graph** consists of a graph $G = (V, E)$ and **weight function** $w : E \to \mathbb{R}$ defined on the edges of G.
The **weight** of a subgraph H of G is the sum of the edges in H.

Example.



Definition. For a graph G, a **spanning tree** T is a subgraph of G which is a tree and contains every vertex of G.

Goal: For a weighted graph G, find a minimum-weight spanning tree.

Kruskal's algorithm

Kruskal's Algorithm finds a minimum-weight spanning tree in a weighted graph.

1. Initialization: Order the edges from lowest to highest weight:

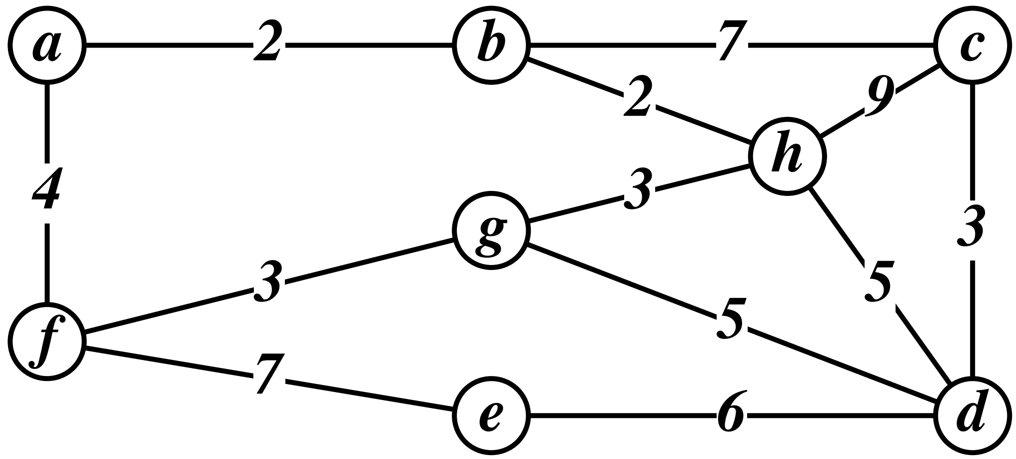
$$w(e_1) \leq w(e_2) \leq w(e_3) \leq \cdots \leq w(e_k).$$

2. Step 1: Define $T = \{e_1\}$ and grow the tree as follows:
3. Step i: Determine if adding e_i to T would create a cycle.
   - If not, add e_i to the set T.
   - If so, do nothing.

   If you have a spanning tree, STOP. You have a m.w.s.t. Otherwise, continue onto step $i + 1$.

Kruskal's algorithm

Example. Run Kruskal's algorithm on the following graph:



Notes on Kruskal's algorithm

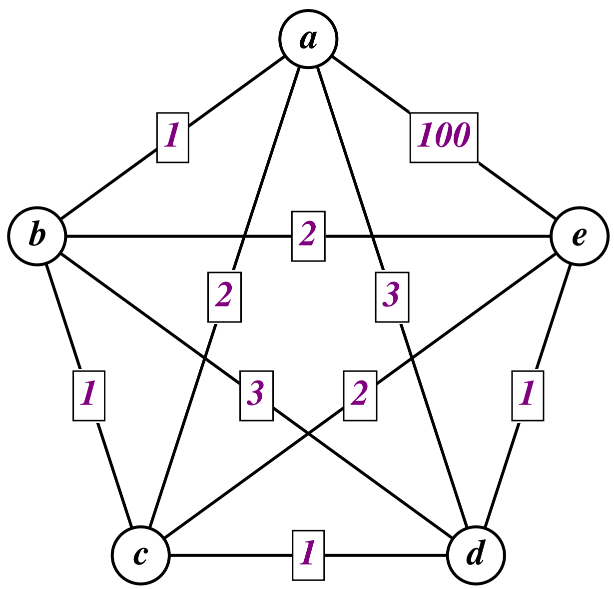
- Proof of correctness similar to homework. Must additionally verify that the spanning tree is indeed minimum-weight.
- Kruskal's algorithm is an example of a *greedy algorithm*. (It chooses the cheapest edge at each point.)
- Greedy algorithms don't always work.

The traveling salesman problem

Motivation: Visit all nodes and return home as cheaply as possible.

- Least cost trip flying between five major cities.
- Optimal routes for delivering mail, collecting garbage.
- Finding a trip to all buildings on campus, return.

Goal: Find a minimum-weight Hamiltonian cycle in a weighted graph.



We can not use a greedy algorithm to find this **TSP tour**!

The traveling salesman problem

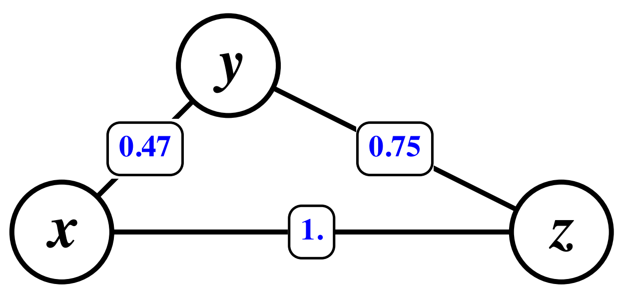
- It is *hard* to find an optimum solution.
- Goal: Create an easy-to-find *pretty good* solution.

Theorem. When the edge weights satisfy the triangle inequality, the *tree shortcut algorithm* finds a tour that costs at most twice the optimum tour.

Recall. The **triangle inequality** says that if x, y, and z are vertices, then $\mathrm{wt}(xy) + \mathrm{wt}(yz) \leq \mathrm{wt}(xz)$.

Example. Euclidean distances

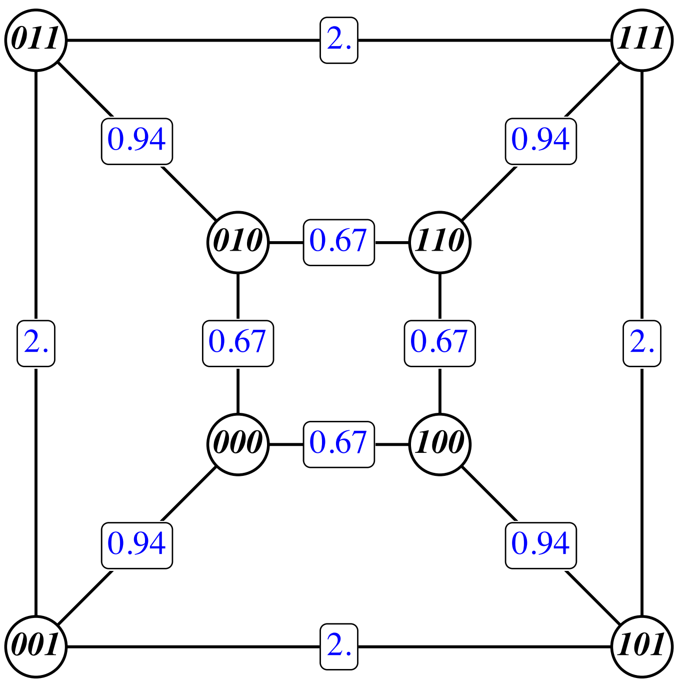
Non-example. Airfares

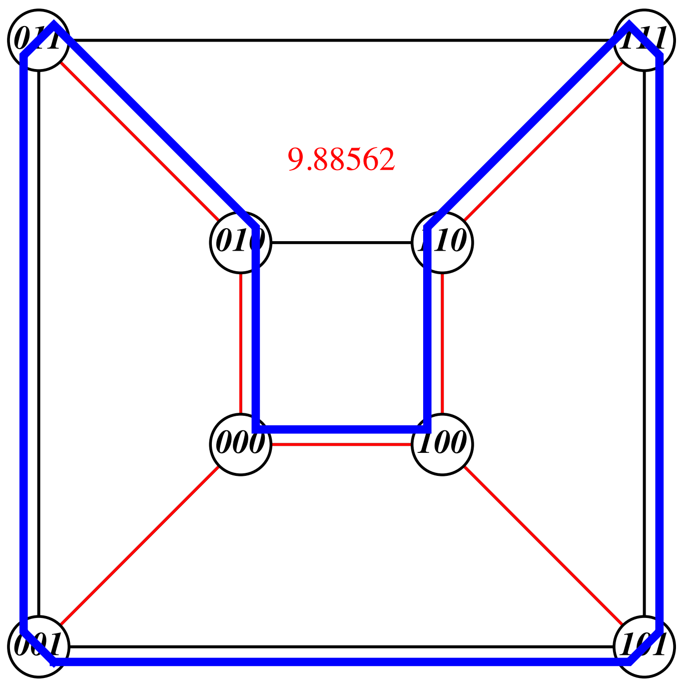


Finding a good TSP-tour

The **Tree Shortcut Algorithm** to find a good TSP-tour

1. Find a minimum-weight spanning tree (Use Kruskal's Algorithm)
2. Walk in a circuit around the edges of the tree.
3. Take shortcuts to find a tour.





Proof of theorem

Theorem. When the edge weights satisfy the triangle inequality, the *tree shortcut algorithm* finds a tour that costs at most twice the optimum tour.

Proof. Define:

- TSP_A: TSP tour from shortcutting spanning tree
- $CIRC_A$: Circuit constructed by doubling spanning tree
- MST: Minimum-weight spanning tree
- TSP^*: Minimum-weight TSP tour

Then,

$$\mathrm{wt}(TSP_A) \leq \mathrm{wt}(CIRC_A) = 2\,\mathrm{wt}(MST) \leq 2\,\mathrm{wt}(TSP^*).$$