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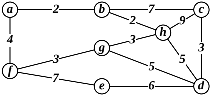
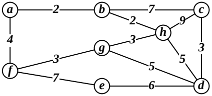
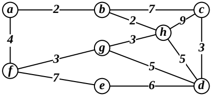
Definition. A **weighted graph** consists of a graph $G = (V, E)$ and **weight function** $w : E \to \mathbb{R}$ defined on the edges of G.
The **weight** of a subgraph H of G is the sum of the edges in H.

Example.







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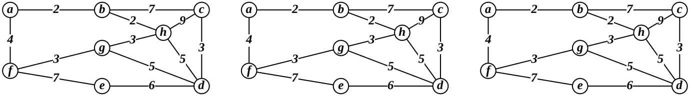

Definition. For a graph G, a **spanning tree** T is a subgraph of G which is a tree and contains every vertex of G.

Goal: For a weighted graph G, find a minimum-weight spanning tree.

Kruskal's algorithm

Kruskal's Algorithm finds a minimum-weight spanning tree in a weighted graph.

1. Initialization: Order the edges from lowest to highest weight:

$$w(e_1) \leq w(e_2) \leq w(e_3) \leq \cdots \leq w(e_k).$$

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   If you have a spanning tree, STOP. You have a m.w.s.t. Otherwise, continue onto step $i + 1$.

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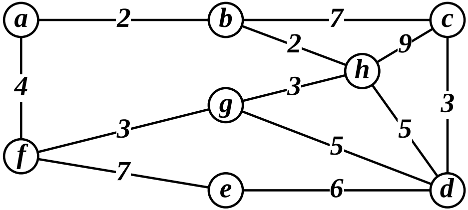

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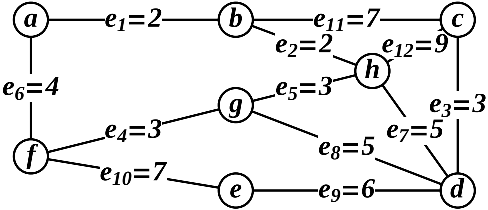

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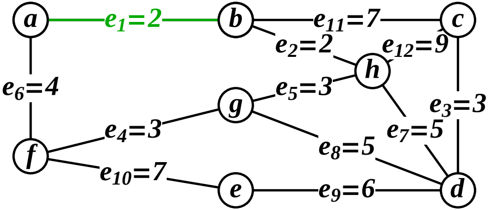

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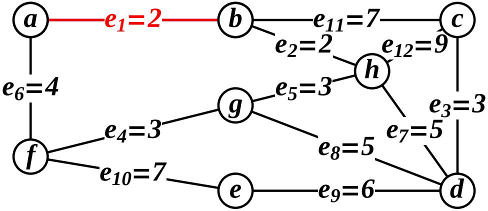

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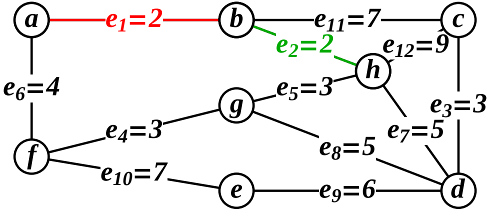

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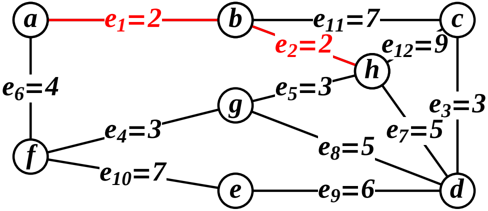

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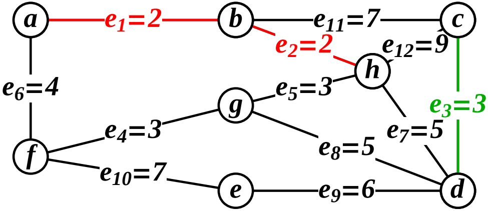

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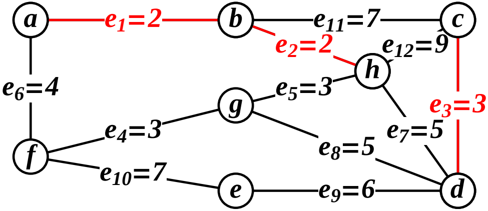

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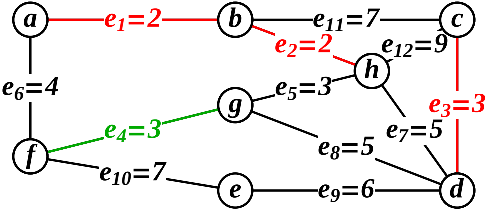

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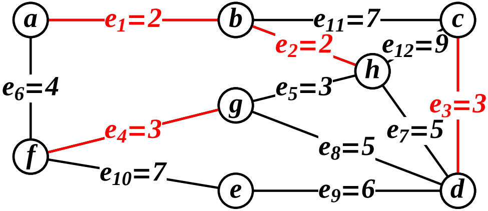

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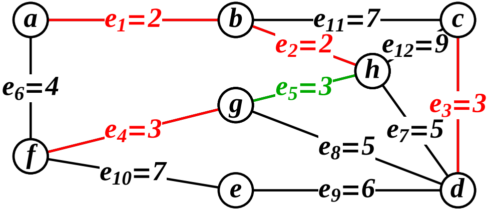

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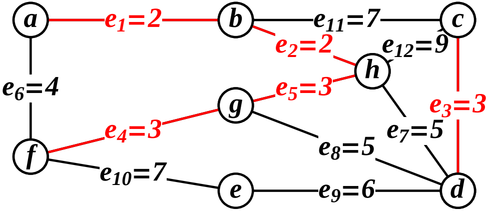

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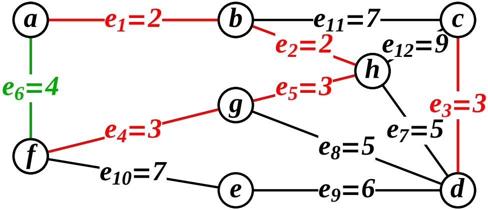

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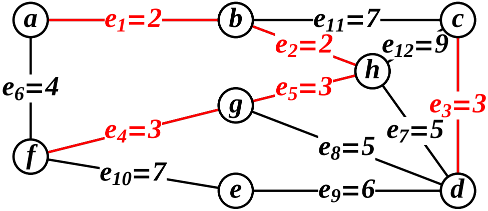

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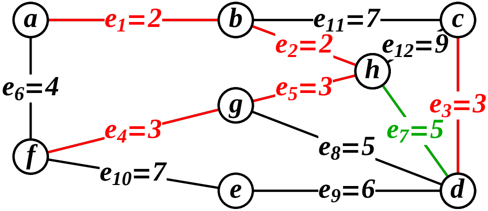

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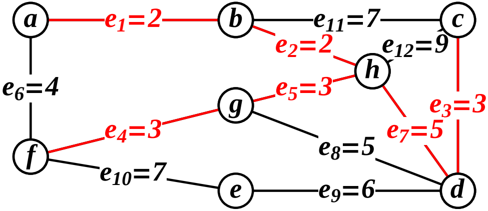

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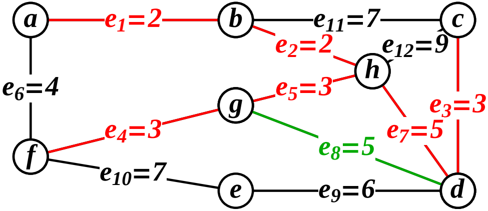

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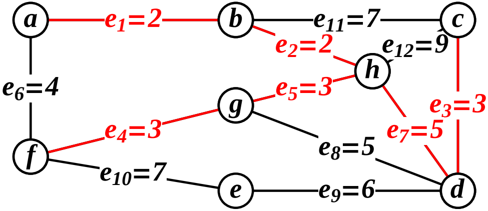

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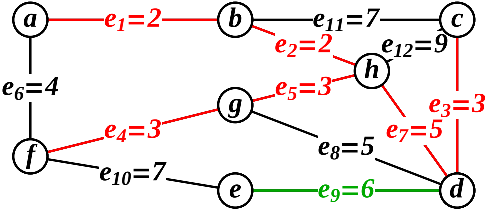

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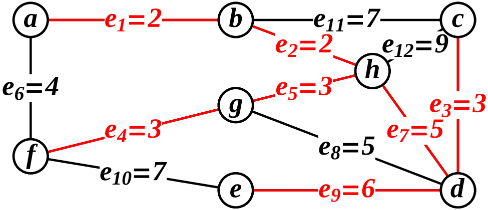

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- Greedy algorithms don't always work.

The traveling salesman problem

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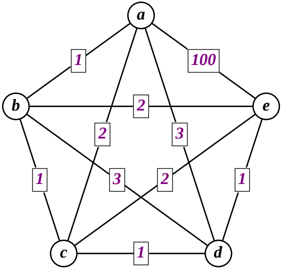

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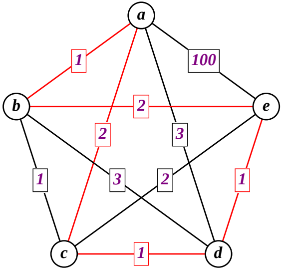

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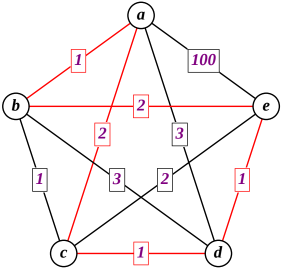

We can not use a greedy algorithm to find this **TSP tour**!

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Recall. The **triangle inequality** says that if x, y, and z are vertices, then $\mathrm{wt}(xy) + \mathrm{wt}(yz) \leq \mathrm{wt}(xz)$.

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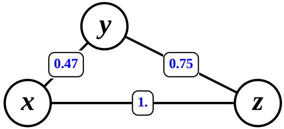
Example. Euclidean distances

Non-example. Airfares



Finding a good TSP-tour

The **Tree Shortcut Algorithm** to find a good TSP-tour

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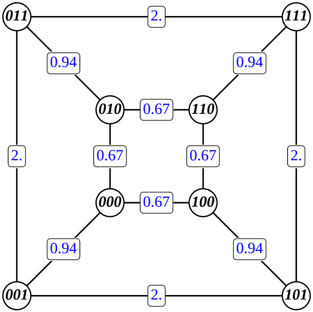

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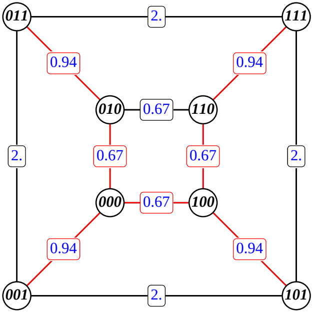

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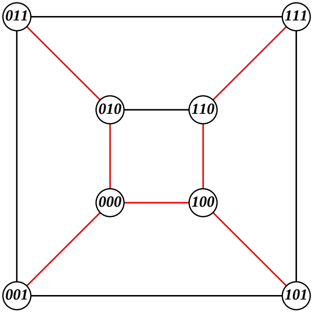

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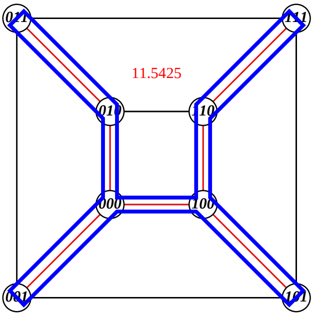

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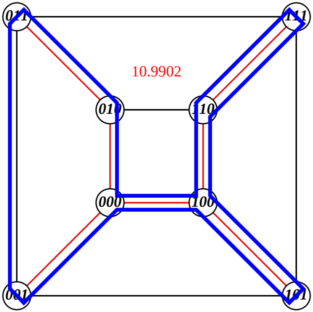

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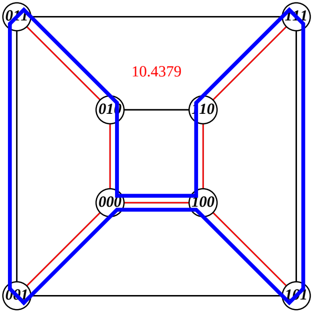

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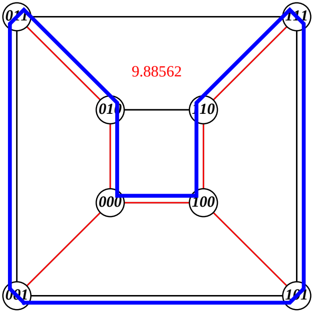

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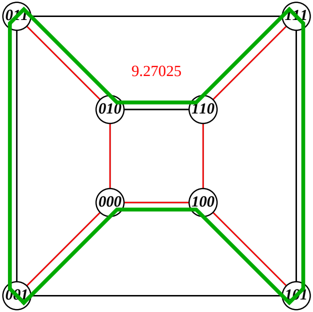

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Then,

$$\mathrm{wt}(TSP_A) \leq \mathrm{wt}(CIRC_A) = 2\,\mathrm{wt}(MST) \leq 2\,\mathrm{wt}(TSP^*).$$