The crossing number of a graph

Some graphs are *almost* planar.

- If $K_{3,3}$ didn't have that last edge, it would be planar!

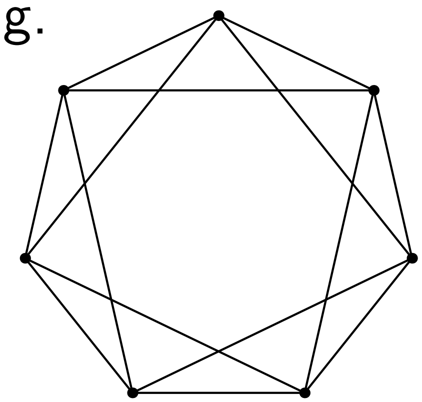
So we ask: How non-planar is it?

★ We will discuss **three** ways to answer this question.

Definition. The **crossing number** of a graph G, denoted $\mathrm{cr}(G)$, is the minimum number of crossings in any simple drawing of G.

- So if G is planar, $\mathrm{cr}(G) = 0$, and if G is non-planar, $\mathrm{cr}(G) \geq 1$.
- To prove $\mathrm{cr}(G) = 1$:
  - Prove G is non-planar (Kuratowski or otherwise) **and**
  - Find a drawing of G with only one crossing.

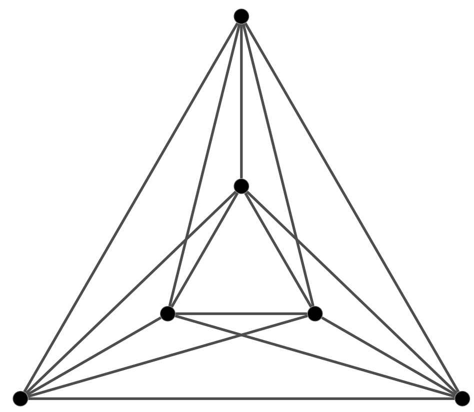
Example.



The crossing number of K_6

Theorem 9.1.4 The crossing number of K_6 is 3.

Proof. First, here is a drawing of K_6 with three crossings:



We conclude that $\mathrm{cr}(K_6) \leq 3$.

Claim: No simple drawing of K_6 has fewer crossings.

- Suppose there exists a drawing of K_6 with two crossings.
- Both crossings involve four distinct vertices.
- Since K_6 has six vertices, there is a vertex v in both crossings.
- If we delete v, the resulting graph would have no crossings.
- This would give a plane drawing of K_5, a contradiction!

Therefore, $\mathrm{cr}(K_6) = 3$.

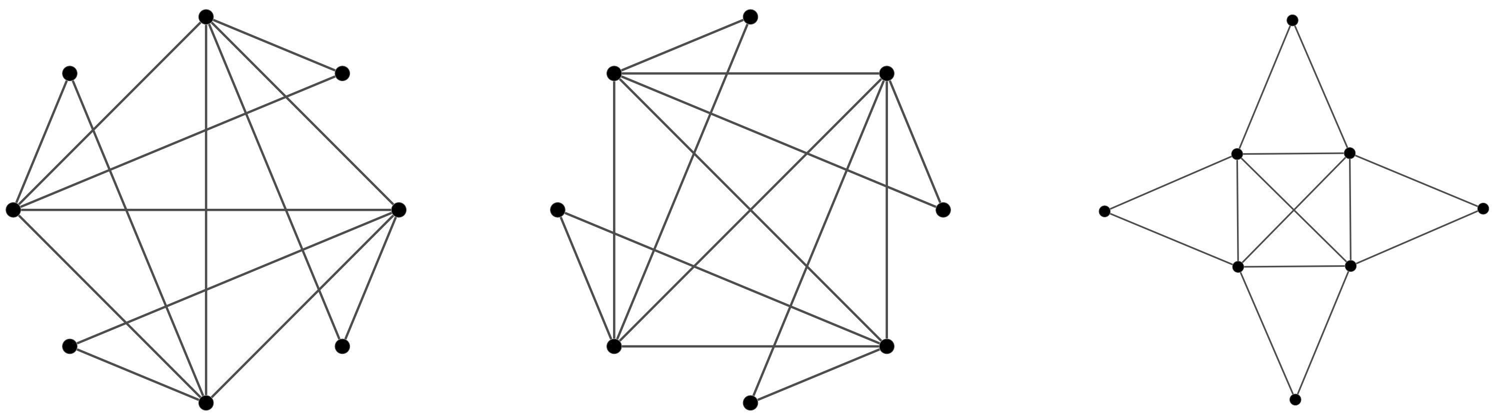
The thickness of a graph

Definition. The **thickness** of a graph G, denoted $\theta(G)$, is the smallest number of planar subgraphs into which G can be decomposed.

That is, find the optimal way to partition of the edge set of G into disjoint subsets, each of which is a planar graph.

- So if G is planar, $\theta(G) = 1$, and if G is non-planar, $\theta(G) \geq 2$.

Example. $\theta(K_8) = 2$ since we know K_8 is nonplanar and below is a decomposition of K_8 into two planar subgraphs:



Theorems about thickness

A simple bound on thickness is:

Theorem 9.2.1. If G has p vertices and q edges, then $\theta(G) \geq \frac{q}{3p-6}$.

Proof. Suppose that $G = H_1 \cup H_2 \cup \cdots \cup H_{\theta(G)}$ is a decomposition of G into planar subgraphs H_i, with p vertices and q_i edges.

We know that each H_i must satisfy $q_i \leq 3p - 6$. Therefore

$$q = \sum_{i=1}^{\theta(G)} q_i \leq \sum_{i=1}^{\theta(G)} (3p-6) = \theta(G)(3p-6).$$

Similarly,

Theorem 9.2.2. If G is a graph with girth ≥ 4, then $\theta(G) \geq \frac{q}{2p-4}$.

Fact: $\theta(K_n) = \begin{cases} \left\lfloor \dfrac{n+7}{6} \right\rfloor & n \neq 9, 10 \\ 3 & n = 9, 10 \end{cases}$ Proved by Beineke, Harary, Vasak, Alekseev, Gonchakov

The genus of a graph

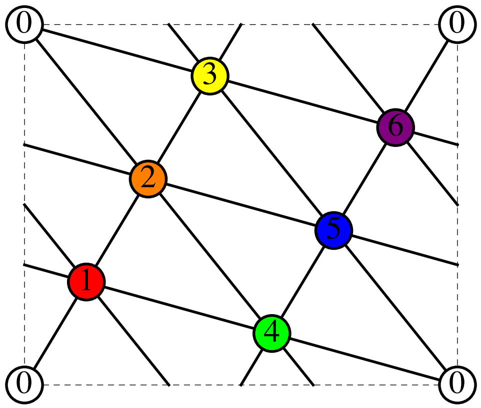
A planar graph can always be **embedded on** a sphere.
That is: it can be drawn without crossings on the surface of a sphere.

Nonplanar graphs can not be embedded on a plane (or sphere).

What about more complicated surfaces? Like a torus?
Example. We can embed K_5 on a torus. (Two ways to see.)

Example. We can even embed K_7 on a torus:

However, we can't embed K_8 on a torus. Perhaps on a surface of genus g?



The genus of a graph

Definition. The **genus** of a graph is the smallest g such that G can be embedded on a surface of genus g with no crossings.

- If G is planar, genus$(G) = 0$; if G is non-planar, genus$(G) \geq 1$.

Fact: (Ringel, Youngs, 1968) The genus of a complete graph is

$$\text{genus}(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$$

Embedding on higher genus surfaces changes Euler's formula!

Theorem. Let G be a graph of genus g. Suppose you have an embedding of G on a surface of genus g with no crossings.
If r is the number of regions, then $p - q + r = \mathbf{2 - 2g}$.

Example. In our embedding of K_5 on the torus (genus 1):

Complete graphs

Planarity statistics for complete graphs:

Statistic	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\mathrm{cr}(K_n)$	0	1	3	9	18	36	60	100	150	225	?	?	?	?	?
$\theta(K_n)$	1	2	2	2	2	**3**	**3**	3	3	3	3	3	3	4	4
$\mathrm{genus}(K_n)$	0	1	1	1	2	3	4	5	6	8	10	11	13	16	18

The crossing number of a complete graph is unknown for $n \geq 13$.

Conjecture. (Guy, 1972) The crossing number of a complete graph is

$$\mathrm{cr}(K_n) = \frac{1}{4}\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

The cases $\mathrm{cr}(K_{11}) = 100$ and $\mathrm{cr}(K_{12}) = 150$ were proved in **2007**.

Theorem. (Zarankiewicz, 1954) There exists a drawing of $K_{m,n}$ with

$$\mathrm{cr}(K_{m,n}) = \frac{1}{4}\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \text{ crossings.}$$