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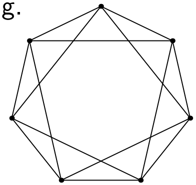

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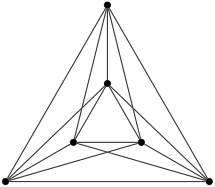

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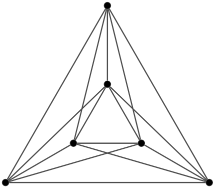

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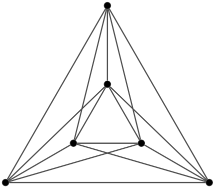

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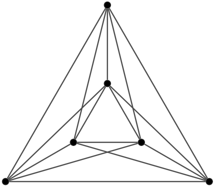

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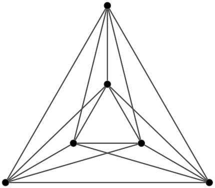

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- This would give a plane drawing of K_5, a contradiction!

Therefore, $\text{cr}(K_6) = 3$.

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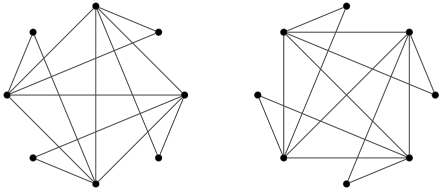

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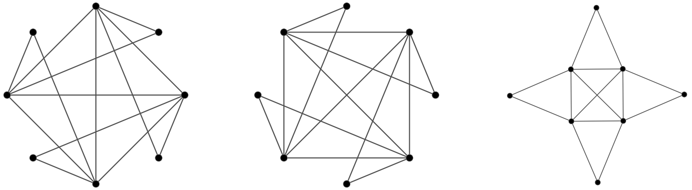

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Fact: $\theta(K_n) = \begin{cases} \left\lfloor \dfrac{n+7}{6} \right\rfloor & n \neq 9, 10 \\ 3 & n = 9, 10 \end{cases}$

Proved by Beineke, Harary, Vasak, Alekseev, Gonchakov

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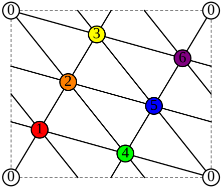

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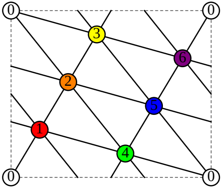

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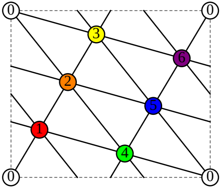

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Embedding on higher genus surfaces changes Euler's formula!

Theorem. Let G be a graph of genus g. Suppose you have an embedding of G on a surface of genus g with no crossings.
If r is the number of regions, then $p - q + r = \mathbf{2 - 2g}$.

Example. In our embedding of K_5 on the torus (genus 1):

Complete graphs

Planarity statistics for complete graphs:

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Theorem. (Zarankiewicz, 1954) There exists a drawing of $K_{m,n}$ with

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