MATH 636, Fall 2014
HOMEWORK 7
due 5:00PM on Tuesday, December 2.

Follow the posted homework guidelines when completing this assignment.
Please **only** consult with your classmates or professor to discuss the problem set.

These four questions are each worth five points each. Continue work on your project.

7-1. Recall that a **Dyck path of length** n is a lattice path from $(0,0)$ to (n,n) that stays above the line $y=x$.)

(a) Find and list the 14 Dyck paths of length 4 and the 14 multiplication schemes for 5 variables.

(b) Use the Catalan bijections from class to determine which Dyck path corresponds to which multiplication scheme.

7-2. Use a bijection to show that sequences $1 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ of length n, where each $a_i \leq i$ are also counted by the Catalan number C_n. For example, when $n=3$, the five sequences are 111, 112, 113, 122, and 123.

[*Hint: Look at the boxes to the left of a Dyck path.*]

7-3. Two combinatorial interpretations of the q-binomial coefficients are given on page 128 of the course notes.

(a) Show that for the permutations π of the multiset $\{1^2, 2^3\}$, $\sum_{\pi \in S_{2,3}} q^{\mathsf{inv}(\pi)} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_q$.

(b) Show that for the set of lattice paths P from $(0,0)$ to $(2,3)$, $\sum_{P \in \mathcal{P}} q^{\mathsf{area}(P)} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_q$.

7-4. Let $\mathcal{C}_n$ denote the set of compositions of n.
For any composition c, define the statistic $\mathsf{parts}(c)$ to be the number of parts of c.
[*In other words, if c is the composition $c_1 + c_2 + \cdots + c_k$, then* $\mathsf{parts}(c) = k$.]

(a) Compute $f_n(q) = \displaystyle\sum_{c \in \mathcal{C}_n} q^{\mathsf{parts}(c)}$.

(b) Use your answer to part (a) to show directly $\lim_{q \to 1} f_n(q) = 2^{n-1}$.

[*Note: We expect part (b) to be true because we know there are 2^{n-1} compositions of n, and part (a) is constructing a q-analog.*]