

Course Notes

Combinatorics, Fall 2014

Queens College, Math 636

Prof. Christopher Hanusa

<http://qcpages.qc.edu/~chanusa/courses/636/14/>

Reference List

The following are books that I recommend to complement this course. I have asked that they be placed *on reserve* in the library.

Benjamin and Quinn. *Proofs that really count.*

Bóna. *A walk through combinatorics.*

Brualdi. *Introductory combinatorics.*

Graham, Knuth, and Patashnik. *Concrete mathematics.*

Mazur. *Combinatorics: A guided tour*

van Lint and Wilson. *A course in combinatorics.*

What is combinatorics?

In this class: Learn how to count ...

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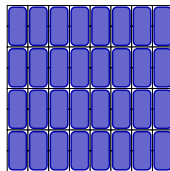
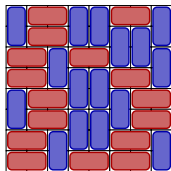
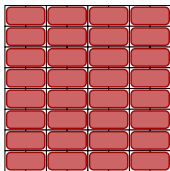
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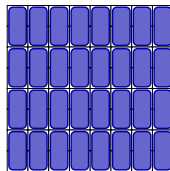
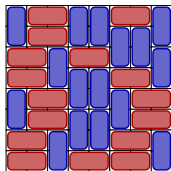
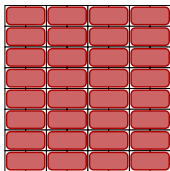
A **domino tiling** is a placement of dominoes on a region, where

- ▶ Each domino covers two squares.
- ▶ The dominoes cover the whole region and do not overlap.

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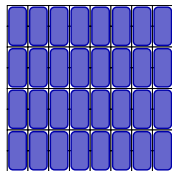
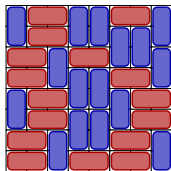
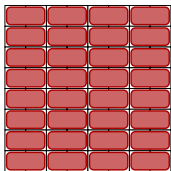
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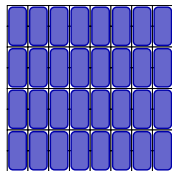
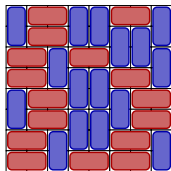
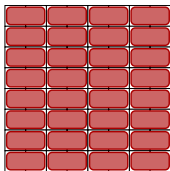
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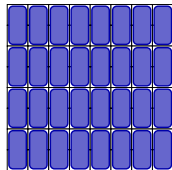
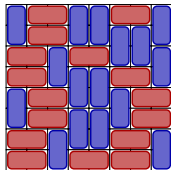
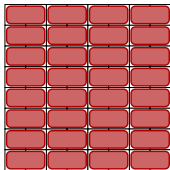
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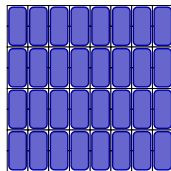
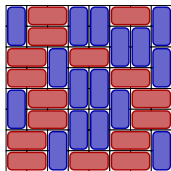
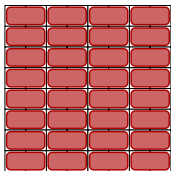
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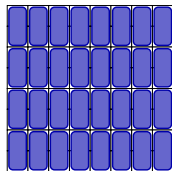
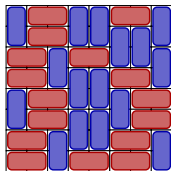
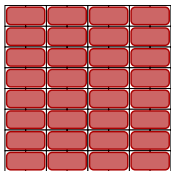
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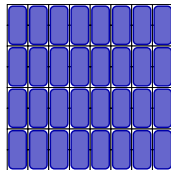
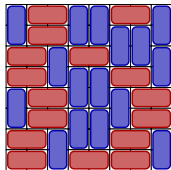
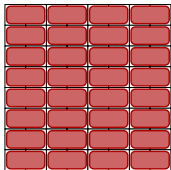
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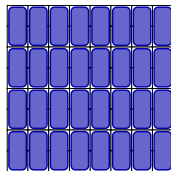
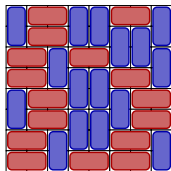
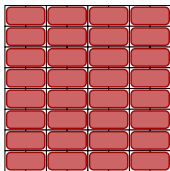
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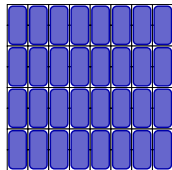
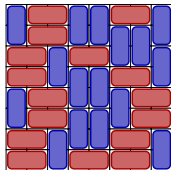
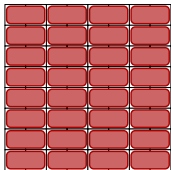
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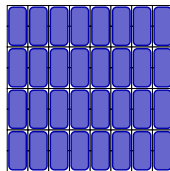
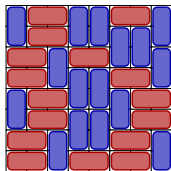
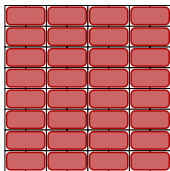
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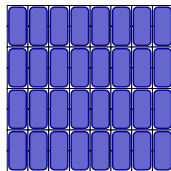
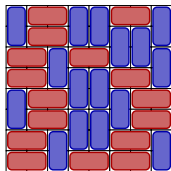
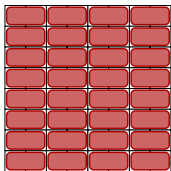


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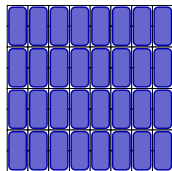
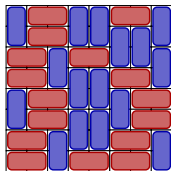
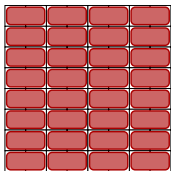
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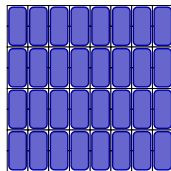
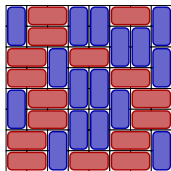
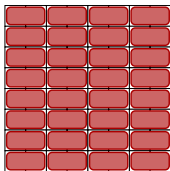
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Domino tilings

How to determine the “answer”?

- ▶ Convert the chessboard into a combinatorial structure (a graph).
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Answer: If m and n are both even, then we have the **formula** (!):

$$\prod_{j=1}^{m/2} \prod_{k=1}^{n/2} \left(4 \cos^2 \frac{\pi j}{m+1} + 4 \cos^2 \frac{\pi k}{n+1} \right).$$

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Combinatorial questions

Given some discrete objects, what properties and structures do they have?

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(Requires many proofs.) (Uses a different kind of reasoning!)

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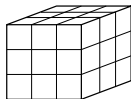
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All homeworks online. Email me by Tuesday, 1st hwk due Thursday.

Cutting a cube

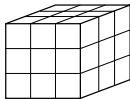
Cut a 3×3 cube into twenty-seven 1×1 cubes using as few cuts as possible. (Rearrangements are allowed.)



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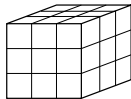


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Conjecture: 6 is the minimum possible number of cuts.

Proof:

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Question: Which 4×4 domino tilings have a *fault line*?

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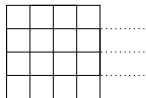


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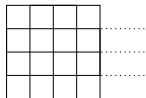


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Every vertical domino must intersect exactly one of these separators; we can count the number of vertical dominoes by adding $x_1 + x_2 + x_3$.

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- ▶ However, a 4×4 chessboard can only hold 8 dominoes, a contradiction!

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 - ▶ Not coverable by dominoes!
- ▶ Suppose there exists a 4×4 domino tiling with no fault line.
 - ▶ Therefore, x_1 , x_2 , and x_3 are all positive (and ≥ 2)
 - ▶ And there must be at least six vertical dominoes.
 - ▶ Similarly, there are at least six horizontal dominoes
- ▶ However, a 4×4 chessboard can only hold 8 dominoes, a contradiction!

Therefore, it is impossible for a 4×4 domino tiling to have no fault lines.

Numbers are everywhere

Arrange yourselves into groups of **four** people,
With people you **don't know**.

- ▶ Introduce yourself. (your name, where you are from)
- ▶ What brought you to this class?
- ▶ Fill out **the front of** your notecard:
 - ▶ Write your name. (Stylize if you wish.)
 - ▶ Write a few words related to your name.
 - ▶ Draw something in the remaining space.
- ▶ Discuss with your groupmates why you wrote what you wrote.
- ▶ Exchange contact information. (phone / email / other)
- ▶ *Small talk suggestion:* Numbers are everywhere.
What number do you identify yourself with and why?

Four Counting Questions (p. 2)

Here are four counting questions.

- Q1. How many 8-character passwords are there using A–Z, a–z, 0–9?
- Q2. In how many ways can a baseball manager order nine fixed baseball players in a lineup?
- Q3. How many Pick-6 lottery tickets are there?
(Choose six numbers between 1–40.)
- Q4. How many possible orders for a dozen donuts are there when the store has 30 varieties?

Think Write Pair Share: Use your powers of **estimation**.

Order the answers from smallest to largest.

_____ < _____ < _____ < _____

The game of Nim

Here are the rules of the two-player game Nim:

- 1 The game starts with two piles of counters.
- 2 Alternating play, each player removes some number of counters from **either** pile.
- 3 The player who removes the last counter wins.

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Let's play!

- ▶ First, get a feel for the game. Try starting with initial piles of $(4,6)$, $(5,5)$, $(3,10)$, and $(7,8)$.
- ▶ Next, start to develop some strategies for winning.
- ▶ Finally, determine conditions under which the first player will always win if she plays optimally, and similarly for the second player.

If you finish this before time is up, try playing Nim with three or more initial piles.