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- ▶ Characterize what solutions look like.
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In general, the number of non-negative integer solutions to $x_1 + x_2 + \cdots + x_n = k$ is _____.

Question: How many **positive** integer solutions are there of $x_1 + x_2 + x_3 + x_4 = 10$, where $x_4 \geq 3$?

The sum principle

Often it makes sense to break down your counting problem into smaller, **disjoint**, and easier-to-count sub-problems.

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This illustrates the **sum principle**:

Suppose the objects to be counted can be broken into k disjoint and exhaustive cases. If there are n_j objects in case j , then there are $n_1 + n_2 + \cdots + n_k$ objects in all.

Counting pitfalls

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 - ▶ Often, **misapplying** the product principle.
 - ▶ **Ask:** Do cases need to be counted in different ways?
 - ▶ **Ask:** Does the same object appear in multiple ways?

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Common example: A deck of cards.

There are four suits: Diamond \diamond , Heart \heartsuit , Club \clubsuit , Spade \spadesuit .

Each has 13 cards: Ace, King, Queen, Jack, 10, 9, 8, 7, 6, 5, 4, 3, 2.

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Example. Suppose you are dealt two diamonds between 2 and 10. In how many ways can the product be even?

Overcounting

Example. In Blackjack you are dealt 2 cards: 1 face-up, 1 face-down. In how many ways can the face-down card be an Ace and the face-up card be a **Heart** ♥?

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Remember to ask: Do cases need to be counted in different ways?

Overcounting

Example. How many 4-lists taken from $[9]$ have at least one pair of adjacent elements equal?

Examples: 1114, 1229, 5555

Non-examples: 1231, 9898.

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Q1: How many 4-lists taken from $[9]$ have **at least one** pair of adjacent elements equal?

—**Compare this to**—

Q2: How many 4-lists taken from $[9]$ have **no** pairs of adjacent elements equal?

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Example. When playing five-card poker, what is the probability that you are dealt a full house?

[*Three cards of one type and two cards of another type.*] 5 5 5 K K

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$$\frac{3744}{2598960} \approx 0.14\%$$

Pascal's triangle

Pascal's identity gives us the recurrence $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.
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$n \setminus k$	0	1	2	3	4	5	6	7
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Seq's in Pascal's triangle:

1, 2, 3, 4, 5, ... $\binom{n}{1}$

($a_n = n$)

1, 3, 6, 10, 15, ... $\binom{n}{2}$

triangular

1, 4, 10, 20, 35, ... $\binom{n}{3}$

tetrahedral

1, 2, 6, 20, 70, ... $\binom{2n}{n}$

centr. binom.

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Online Encyclopedia of Integer Sequences:

<http://oeis.org/>

Binomial Theorem

Theorem 2.2.2. Let n be a positive integer. For all x and y ,

$$(x + y)^n = x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + y^n.$$

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Rewrite in summation notation!

Determine the generic term $[\binom{n}{k}x^k y^{n-k}]$ and the bounds on k

$$(x + y)^n = \sum$$

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From the n factors $(x + y)$, you must choose a “ y ” exactly k times. Therefore, $\binom{n}{k}$ ways. We recover the desired equation. \square