Introduction to Bijections

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Set A: $\{\ \emptyset,\ \{s_1\}, \{s_2\}, \{s_1, s_2\}, \{s_3\}, \{s_1, s_3\}, \{s_2, s_3\}, \{s_1, s_2, s_3\}\}$

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Rule: Given $a \in A$, (a is a subset), define $b \in B$ (b is a word):
If $s_i \in a$, then letter i in b is 1. If $s_i \notin a$, then letter i in b is 0.

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Difficulties:

- Finding the function or rule (requires rearranging, ordering)
- Proving the function or rule (show it **IS** a bijection).

What is a Function?

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Example. Let A be the set of 3-subsets of $[n]$ and let B be the set of 3-lists of $[n]$. Then define $f : A \to B$ to be the function that takes a 3-subset $\{i_1, i_2, i_3\} \in A$ (with $i_1 \leq i_2 \leq i_3$) to the word $i_1 i_2 i_3 \in B$.

Question: Is $\text{rng}(f) = B$?

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What is an example of a function that is onto and not one-to-one?

Proving a Bijection

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Step 1: Find a candidate bijection.

Strategy. Try out a small (enough) example. Try $n = 5$ and $k = 2$.

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Guess: Let S be a k-subset of $[n]$. Perhaps $f(S) =$ ______.

Proving a Bijection

Step 2: Prove f is well defined.

The function f is well defined. If S is any k-subset of $[n]$, then S^c is a subset of $[n]$ with $n - k$ members. Therefore $f : A \to B$.

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f is 1-to-1: Suppose that S_1 and S_2 are two k-subsets of $[n]$ such that $f(S_1) = f(S_2)$. That is, $S_1^c = S_2^c$. This means that for all $i \in [n]$, then $i \notin S_1$ if and only if $i \notin S_2$. Therefore $S_1 = S_2$ and f is 1-to-1.

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f is onto: Suppose that $T \in B$ is an $(n - k)$-subset of $[n]$.
We must find a set $S \in A$ satisfying $f(S) = T$. Choose $S =$ ______.
Then $S \in A$ (why?), and $f(S) = S^c = T$, so f is onto.

We conclude that f is a bijection and therefore, $\binom{n}{k} = \binom{n}{n-k}$.

Using the Inverse Function

When $f : A \to B$ is 1-to-1, we can define f's **inverse**.

We write f^{-1}, and it is a function from rng(f) to A.

It is defined via f. If $f : a \mapsto b$, then $f^{-1} : b \mapsto a$.

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Theorem. Suppose that A and B are finite sets and that $f : A \to B$ is a function. If f^{-1} is a function with domain B, then f is a bijection.

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Proof. Since f^{-1} is only defined when f is 1-to-1, we need only prove that f is onto. Suppose $b \in B$. By assumption, $f^{-1}(b) \in A$ exists and $f(f^{-1}(b)) = b$. So f is onto, and is a bijection.

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Consequence: An alternative method for proving a bijection is:

- Find a rule $g : B \to A$ which always takes $f(a)$ back to a.
- Verify that the domain of g is *all of B*.

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Proof. Let A be the set of even-sized subsets of $[n]$ and let B be the set of odd-sized subsets of $[n]$. Consider the function

$$f(S) = \begin{cases} S - \{1\} & \text{if } 1 \in S \\ S \cup \{1\} & \text{if } 1 \notin S \end{cases}.$$

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Eyebrow-Raising Consequence: $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$