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Definition: A **combinatorial proof** of an identity $X = Y$ is a **proof by counting (!)**. You find a set of objects that can be interpreted as a combinatorial interpretation of both the **left hand side (LHS)** and the **right hand side (RHS)** of the equation. As both sides of the equation count the same set of objects, they must be equal!

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- ▶ It is important to get the set of objects right.
- ▶ To do this, you must ask a good question: “In how many ways...”

A Simple Combinatorial Proof

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Because the two quantities count the same set of objects in two different ways, the two answers are equal. □

Another Simple Combinatorial Proof

Example. Prove *Equation (2.4)*: $k \binom{n}{k} = n \binom{n-1}{k-1}$.

Analytic Proof:

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Analytic Proof:

Combinatorial Proof:

Question: In how many ways can we choose from n club members a committee of k members with a chairperson?

Answer 1:

Answer 2:

Because the two quantities count the same set of objects in two different ways, the two answers are equal. □

Pascal's Identity

Example. Prove *Theorem 2.2.1*: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Combinatorial Proof:

Question: In how many ways can we choose k flavors of ice cream if n different choices are available?

Answer 1:

Answer 2:

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Summing Binomial Coefficients

Example. Prove *Equation (2.3)*: $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$.

Analytic Proof: ???

Combinatorial Proof:

Question: How many subsets of $\{1, 2, \dots, n\}$ are there?

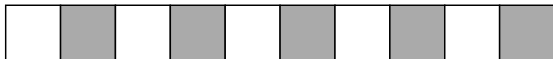
Answer 1: Condition on how many elements are in a subset.

Answer 2:

Because the two quantities count the same set of objects in two different ways, the two answers are equal. □

Tiling a board with dominos and squares

Question: How many ways are there to tile a $1 \times n$ board using only dominos and squares?



Definition: Let $f_n = \#$ of ways to tile a $1 \times n$ board.

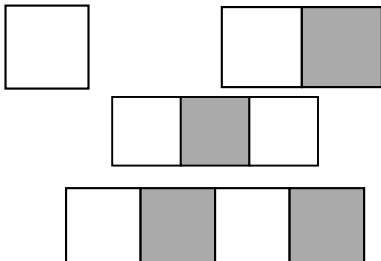
$$f_0 = 1$$

$$f_1 =$$

$$f_2 =$$

$$f_3 =$$

$$f_4 =$$



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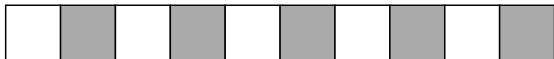
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$$f_2 = 2$$

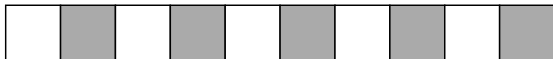
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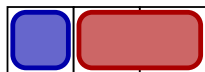
$$f_0 = 1$$

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$$f_2 = 2$$

$$f_3 = 3$$

$$f_4 =$$



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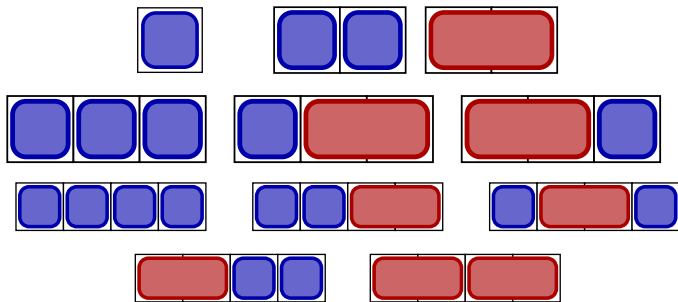
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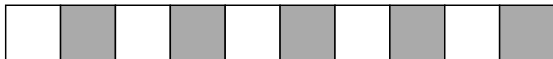
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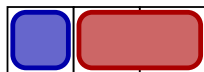
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Fibonacci!

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Fibonacci numbers f_n satisfy

▶ $f_0 = f_1 = 1$

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There are f_n tilings of a $1 \times n$ board

Every tiling ends in either:

▶ a square



▶ a domino



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▶ **How many?** Fill the initial $1 \times (n - 1)$ board in f_{n-1} ways.



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▶ **How many?** Fill the initial $1 \times (n-2)$ board in f_{n-2} ways.

Total: $f_{n-1} + f_{n-2}$

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Fibonacci numbers f_n satisfy

- ▶ $f_0 = f_1 = 1$ ✓
- ▶ $f_n = f_{n-1} + f_{n-2}$ ✓

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- ▶ **How many?** Fill the initial $1 \times (n - 1)$ board in f_{n-1} ways.

- ▶ a domino



- ▶ **How many?** Fill the initial $1 \times (n - 2)$ board in f_{n-2} ways.

Total: $f_{n-1} + f_{n-2}$

Fibonacci identities

We have a new definition for Fibonacci:

f_n = the number of square-domino tilings of a $1 \times n$ board.

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► Did you know that $f_{2n} = (f_n)^2 + (f_{n-1})^2$?

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f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}
1	2	3	5	8	13	21	34	55	89	144	233	377	610

$$f_8 = f_4^2 + f_3^2$$

$$34 = 25 + 9$$

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$$f_{14} = f_7^2 + f_6^2$$

$$610 = 441 + 169$$

Proof that $f_{2n} = (f_n)^2 + (f_{n-1})^2$

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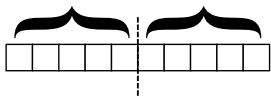
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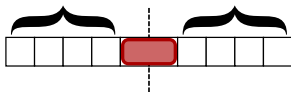
Answer 1. Duh, f_{2n} .

Answer 2. Ask whether there is a break in the middle of the tiling:

Either there is...



Or there isn't...

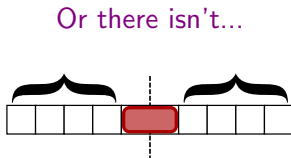
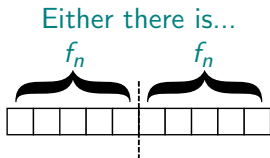


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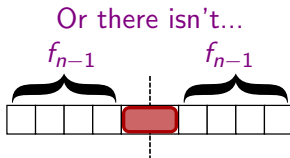
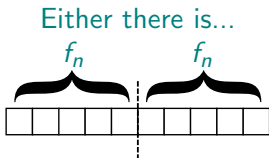


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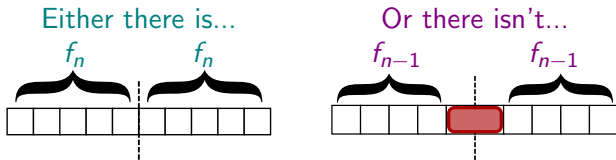


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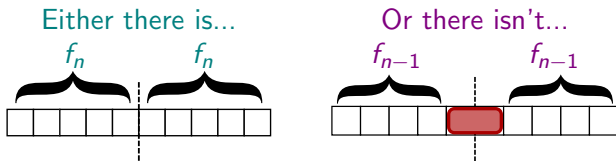
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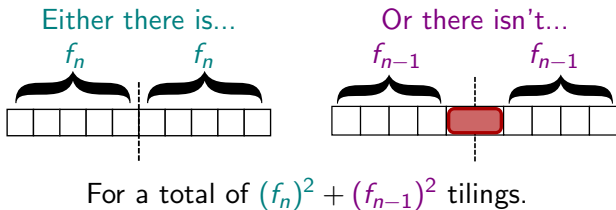
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Further reading:

-  Arthur T. Benjamin and Jennifer J. Quinn
Proofs that Really Count, MAA Press, 2003.