mmooRREE COUNTING!

Question: In how many ways can we place k objects in n boxes?

Answer: It depends.

- What do the objects look like?
  - Do the objects all look the same?
- What do the boxes look like?
  - Do the boxes all look the same?
- Are there any restrictions?
  - Is there a size limit?
  - Must there be an object in each box?

Counting distributions

Definition: A **distribution** is an assignment of objects to recipients.

Certain counting problems can be revisited in this framework:

$$\left\{ \begin{array}{c} \text{Five-letter passwords} \\ \text{on } \{A, B, C, D, E, F, G\} \end{array} \right\} \text{ correspond to } \left\{ \begin{array}{c} \text{Distributions of} \\ \underline{\quad\quad} \text{ distinct objects} \\ \text{into } \underline{\quad\quad} \text{ distinct boxes} \end{array} \right\}$$

- What are candidates for objects, boxes?
- View as a function
- View as a distribution
- Find the restriction

THE CHART

Question: In how many ways can we place k objects in n boxes?

Distributions of		Restrictions on # objects received			
k objects	**n boxes**	none	≤ 1	≥ 1	$= 1$
distinct	distinct				
identical	distinct				
distinct	identical				
identical	identical				

Where do our known answers fit into the table? (Use function view)

- n^k: Objects distinct, Boxes distinct, no restriction.
- $(n)_k$: Objects distinct, Boxes distinct, ≤ 1 object per box.
- $n!$: Permutations. What about when $n \neq k$?
- $\binom{n}{k}$: Objects ________, Boxes _________, ___________________.
- $\left(\!\binom{n}{k}\!\right)$: Objects ________, Boxes _________, ___________________.

THE CHART

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Distributions of		Restrictions on # objects received			
k **objects**	n **boxes**	none	≤ 1	≥ 1	$= 1$
distinct	distinct				
identical	distinct				
distinct	identical				
identical	identical				

We can also fill in these answers:

- Objects identical, Boxes distinct, ≥ 1 object per box:

- Objects identical, Boxes distinct, $= 1$ object per box:

Distinct objects in indistinguishable boxes

When placing k distinguishable objects into n indistinguishable boxes, what matters? ____________________________________

- Each object needs to be in some box.
- No object is in two boxes.

We have rediscovered ____________________.

So ask "How many set partitions are there of a set with k objects?"

Or even, "How many set partitions are there of k objects into n parts?"

Stirling numbers

The **Stirling number of the second kind** counts the number of ways to partition a set of k elements into i **non-empty** subsets.

Notation: $S(k,i)$ or $\left\{ {k \atop i} \right\}$. $\leftarrow$ **Careful about this order!**

k	$\left\{ {k \atop 0} \right\}$	$\left\{ {k \atop 1} \right\}$	$\left\{ {k \atop 2} \right\}$	$\left\{ {k \atop 3} \right\}$	$\left\{ {k \atop 4} \right\}$	$\left\{ {k \atop 5} \right\}$	$\left\{ {k \atop 6} \right\}$	$\left\{ {k \atop 7} \right\}$
0	1							
1		1						
2		1	1					
3		1	3	1				
4		1	7	6	1			
5		1	15	25	10	1		
6		1	31	90	65	15	1	
7		1						1

In Stirling's triangle:

$S(k,1) = S(k,k) = 1.$

$S(k,2) = 2^{k-1} - 1.$

$S(k,k-1) = \binom{k}{2}.$

Later: Formula for $S(k,i)$.

To fill in the table, find a recurrence for $S(k,i)$:

Ask: In how many ways can we place k objects into i boxes?

We'll condition on the placement of element $\#i$:

THE CHART

Question: In how many ways can we place k objects in n boxes?

Distributions of		Restrictions on # objects received			
k **objects**	n **boxes**	none	≤ 1	≥ 1	$= 1$
distinct	distinct	n^k	$(n)_k$		$n!$ or 0
identical	distinct	$\left(\!\binom{n}{k}\!\right)$	$\binom{n}{k}$	$\left(\!\binom{n}{k-n}\!\right)$	1 or 0
distinct	identical				
identical	identical				

$S(k, n)$ counts ways to place k distinct obj. into n identical boxes.

What if we then label the boxes?
(*Note that here we have counted onto functions* $[k] \to [n]$.)

How many ways to distribute distinct objects into identical boxes:

- If there is exactly one item in each box?
- If there is at most one item in each box?
- What about with no restrictions? $(n \geq k \rightsquigarrow$ Bell number $B_k)$

Bell numbers

Definition: The **Bell number** B_k is the number of partitions of a set with k elements, into any number of non-empty parts.

We have $B_k = S(k,0) + S(k,1) + S(k,2) + \cdots + S(k,k)$.

B_0	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9
1	1	2	5	15	52	203	877	4140	21147

Theorem 2.3.3. The Bell numbers satisfy a recurrence:

$$B_k = \binom{k-1}{0} B_0 + \binom{k-1}{1} B_1 + \cdots + \binom{k-1}{k-1} B_{k-1}.$$

Proof: How many partitions of $\{1, \ldots, k\}$ are there?

LHS: B_k, obviously.

RHS: Condition on the box containing the last element k:
How many partitions of $[k]$ contain i elements in the box with k?

Indistinguishable objects in indistinguishable boxes

When placing k indistinguishable objects into n indistinguishable boxes, what matters? ____________________

- We are partitioning the **integer** k instead of the **set** $[k]$.

Example. What are the partitions of 6?

Definition: $P(k, i)$ is the number of partitions of k into i parts.

Example. We saw $P(6,1) = 1$, $P(6,2) = 3$, $P(6,3) = 3$, $P(6,4) = 2$, $P(6,5) = 1$, and $P(6,6) = 1$.

Definition: $P(k)$ is the number of partitions of k into any number of parts.

Example. $P(6) = 1 + 3 + 3 + 2 + 1 + 1 = 11$.

THE CHART, COMPLETED

Question: In how many ways can we place k objects in n boxes?

Distributions of		Restrictions on # objects received			
k **objects**	n **boxes**	none	≤ 1	≥ 1	$= 1$
distinct	distinct	n^k	$(n)_k$	$n!S(k,n)$	$n!$ or 0
identical	distinct	$\left(\!\binom{n}{k}\!\right)$	$\binom{n}{k}$	$\left(\!\binom{n}{k-n}\!\right)$	1 or 0
distinct	identical	$\sum S(k,i)$	1 or 0	$S(k,n)$	1 or 0
identical	identical	$\sum P(k,i)$	1 or 0	$P(k,n)$	1 or 0

$P(k,n)$ counts ways to place k identical obj. into n identical boxes.

How many ways to distribute identical objects into identical boxes:

- If there is exactly one item in each box?
- If there is at most one item in each box?
- What about with no restrictions?

(This is the # of integer partitions of k into at most n parts.)