

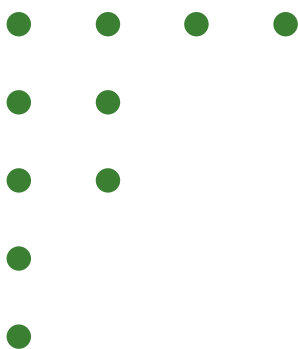
More about partitions

- ▶ $3 + 1 + 1$, $1 + 3 + 1$, and $1 + 1 + 3$ are all the same partition, so we will write the numbers in non-increasing order.
- ▶ We use greek letters to denote partitions, often λ (“lambda”), μ (“mu”), and ν (“nu”).
- ▶ We’ll write: $\lambda : n = n_1 + n_2 + \cdots + n_k$ or $\lambda \vdash n$.

For example, $\lambda : 5 = 3 + 1 + 1$, or $\lambda = 311$, or $\lambda = 3^1 1^2$, or $311 \vdash 5$.

A pictorial representation of $\lambda = n_1 n_2 \cdots n_k$ is its *Ferrers diagram*, a left-justified array of dots with k rows, containing n_i dots in row i .

Example. The Ferrers diagram of $42211 \vdash 10$ is



The **conjugate** of a partition λ is the partition λ^c which interchanges rows and columns.

Some partitions are **self-conjugate**, satisfying $\lambda = \lambda^c$.

A generating function for partitions

Recall from our basketball example: The generating function for the number of ways to partition an integer into parts of size 1, 2, or 3 is

$$\frac{1}{(1-x)} \frac{1}{(1-x^2)} \frac{1}{(1-x^3)}$$

If we include parts of any size, we infer:

Let $P(n)$ be the number of partitions of the integer n . Then

$$\sum_{n \geq 0} P(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

Notes:

- ▶ Infinite product! But, for any n only finitely many terms involved.
- ▶ There is a beautiful generating function, but **no nice formula!**
- ▶ Finding a generating function for a subset of partitions is easy if you understand each factor in the product.

A formula for integer partitions

Bruinier and Ono (2011) found an algebraic formula for the partition function $P(n)$ as a finite sum of algebraic numbers as follows. Define the weight-2 meromorphic modular form $F(z)$ by

$$F(z) = \frac{1}{2} \frac{E_2(z) - 2E_2(2z) - 3E_2(3z) + 6E_2(6z)}{\eta^2(z)\eta^2(2z)\eta^2(3z)\eta^3(6z)}, \quad (27)$$

where $q = e^{2\pi iz}$, $E_2(q)$ is an Eisenstein series, and $\eta(q)$ is a Dedekind eta function. Now define

$$R(z) = -\left(\frac{1}{2\pi i} \frac{d}{dz} + \frac{1}{2\pi y}\right) F(z), \quad (28)$$

where $z = x + iy$. Additionally let Q_n be any set of representatives of the equivalence classes of the integral binary quadratic form $Q(x, y) = ax^2 + bxy + cy^2$ such that $6 \mid a$ with $a > 0$ and $b \equiv 1 \pmod{12}$, and for each $Q(x, y)$, let α_Q be the so-called CM point in the upper half-plane, for which $Q(\alpha_Q, 1) = 0$. Then

$$P(n) = \frac{\text{Tr}(n)}{24n-1}, \quad (29)$$

where the trace is defined as

$$\text{Tr}(n) = \sum_{Q \in Q_n} R(\alpha_Q). \quad (30)$$

Weisstein, Eric W. "Partition Function P."
 From MathWorld—A Wolfram Web Resource.
<http://mathworld.wolfram.com/PartitionFunctionP.html>

Theorem. $P(n, 2) = \underline{\hspace{2cm}}$.

Partitions: odd parts and distinct parts

Example. THE FOLLOWING AMAZING FACT!!!!1!!!!11!!

$$\boxed{\text{The number of partitions of } n \text{ using only odd parts, } o_n} = \boxed{\text{The number of partitions of } n \text{ using distinct parts, } d_n}$$

Investigation: Does this make sense? For $n = 6$,
 o_6 : d_6 :

Solution. Determine the generating functions

$$O(x) = \sum_{n \geq 0} o_n x^n$$

$$D(x) = \sum_{n \geq 0} d_n x^n$$

See, I told you they were equal. \square

A recurrence relation for $P(n, k)$ (p.78)

Example. Prove a recurrence relation for $P(n, k)$:

$$P(n, k) = P(n - 1, k - 1) + P(n - k, k)$$

Question: How many partitions of n are there into k parts?

LHS: $P(n, k)$

RHS: Condition on whether the smallest part is of size 1.

► **If so,** there are $P(n - 1, k - 1)$ partitions via the bijection

$$f : \left\{ \begin{array}{l} \text{partitions of } n \text{ into } k \text{ parts} \\ \text{with smallest part } 1. \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{partitions of } n - 1 \\ \text{into } k - 1 \text{ parts.} \end{array} \right\}.$$

► **If not:** there are $P(n - k, k)$ partitions via the bijection

$$g : \left\{ \begin{array}{l} \text{partitions of } n \text{ into } k \text{ parts} \\ \text{with smallest part } \neq 1. \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{partitions of } n - k \\ \text{into } k \text{ parts.} \end{array} \right\}.$$

Using conjugation

Theorem 4.4.1. $P(n, k)$ equals $P(n, \text{largest part} = k)$

Proof. The conjugation function $f : \lambda \rightarrow \lambda^c$ is a bijection

$$f : \left\{ \begin{array}{l} \text{partitions of } n \\ \text{into exactly } k \text{ parts} \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{partitions of } n \text{ with} \\ \text{largest part of size } k. \end{array} \right\}.$$

The same bijection gives:

Theorem 4.4.2. _____ equals $P(n, \text{largest part} \leq k)$.

Characterization of self-conjugate partitions

Theorem 4.4.3. $P(n, \text{self conjugate}) = P(n, \text{distinct odd parts})$

Proof. Define a bijection which “unfolds” self-conjugate partitions:

$$f : \left\{ \begin{array}{l} \text{self-conjugate} \\ \text{partitions } \lambda \text{ of } n \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{partitions } \mu \text{ of } n \text{ into} \\ \text{distinct odd parts} \end{array} \right\}.$$

- ▶ Define parts of μ by **unpeeling** λ layer by layer.
- ▶ Iteratively remove the first row and first column of λ .

Question: Is f well defined?

Define the inverse function $g = f^{-1} : \mu \mapsto \lambda$:

- ▶ Find the **center dot** of each part μ_i .
- ▶ **Fold** each μ_i about its center dot.
- ▶ **Nest** these folded parts to create λ .

Question: Is g well defined?

Question: Is $g(f(\lambda)) = \lambda$?

Standard Young Tableaux

Related to some current lines of research in algebra and combinatorics:

A **Young diagram** is a representation of a partition using left-justified boxes.

A **standard Young tableau** is a placement of the integers 1 through n into the boxes, where the numbers in both the rows and the columns are increasing.

The **hook length** $h(i, j)$ of a cell (i, j) is the number of cells in the “hook” to the left and down.

Question: How many SYT are there of shape $\lambda \vdash n$?

Answer:
$$\frac{n!}{\prod_{(i,j) \in \lambda} h(i,j)}$$

