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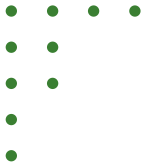

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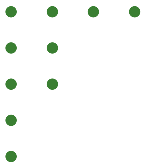

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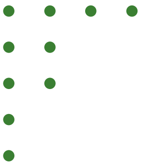

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Some partitions are **self-conjugate**, satisfying $\lambda = \lambda^c$.

A generating function for partitions

Recall from our basketball example: The generating function for the number of ways to partition an integer into parts of size 1, 2, or 3 is

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Notes:

- Infinite product! But, for any n only finitely many terms involved.
- There is a beautiful generating function, but **no nice formula**!

A formula for integer partitions

Bruinier and Ono (2011) found an algebraic formula for the partition function $P(n)$ as a finite sum of algebraic numbers as follows. Define the weight-2 meromorphic modular form $F(z)$ by

$$F(z) = \frac{1}{2}\,\frac{E_2(z) - 2E_2(2z) - 3E_2(3z) + 6E_2(6z)}{\eta^2(z)\,\eta^2(2z)\,\eta^2(3z)\,\eta^3(6z)}, \tag{27}$$

were $q = e^{2\pi i z}$, $E_2(q)$ is an Eisenstein series, and $\eta(q)$ is a Dedekind eta function. Now define

$$R(z) = -\left(\frac{1}{2\pi i}\frac{d}{dz} + \frac{1}{2\pi y}\right)F(z), \tag{28}$$

where $z = x + iy$. Additionally let Q_n be any set of representatives of the equivalence classes of the integral binary quadratic form $Q(x, y) = ax^2 + bxy + cy^2$ such that $6 \mid a$ with $a > 0$ and $b \equiv 1 \pmod{12}$, and for each $Q(x, y)$, let α_Q be the so-called CM point in the upper half-plane, for which $Q(\alpha_Q, 1) = 0$. Then

$$P(n) = \frac{\mathrm{Tr}(n)}{24n - 1}, \tag{29}$$

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Theorem. $P(n, 2) = \underline{\qquad\quad}$.

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See, I told you they were equal. □

A recurrence relation for $P(n, k)$ (p.78)

Example. Prove a recurrence relation for $P(n, k)$:

$$P(n, k) = P(n - 1, k - 1) + P(n - k, k)$$

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- If so, there are $P(n-1, k-1)$ partitions via the bijection

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The same bijection gives:

Theorem 4.4.2. ____________________ equals $P(n, \textit{largest part} \leq k)$.

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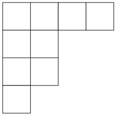
Standard Young Tableaux

Related to some current lines of research in algebra and combinatorics:

A **Young diagram** is a representation of a partition using left-justified boxes.



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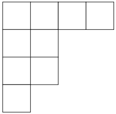
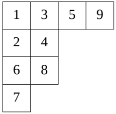


A **standard Young tableau** is a placement of the integers 1 through n into the boxes, where the numbers in both the rows and the columns are increasing.



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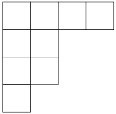

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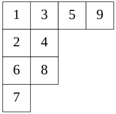

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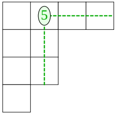

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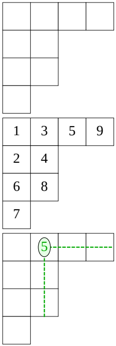

Question: How many SYT are there of shape $\lambda \vdash n$?

Answer: $\dfrac{n!}{\prod_{(i,j)\in\lambda} h(i,j)}$