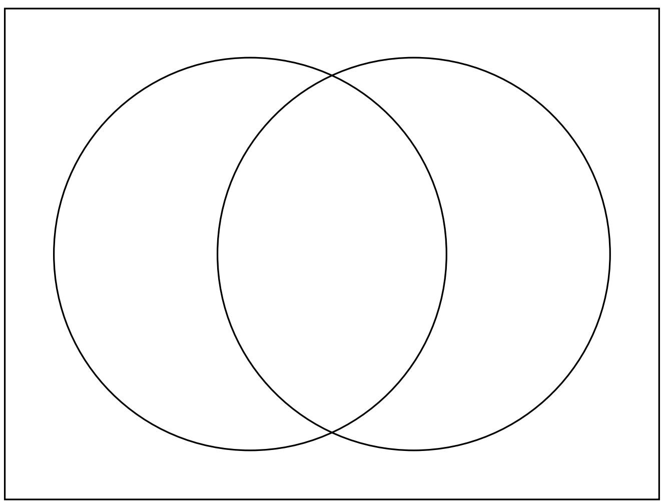
Principle of Inclusion-Exclusion

Example. Suppose that in this class, 14 students play soccer and 11 students play basketball. How many students play a sport?

Solution.

Let S be the set of students who play soccer
and B be the set of students who play basketball.
Then, $|S \cup B| = |S| + |B|$________.



Principle of Inclusion-Exclusion

When $A = A_1 \cup \cdots \cup A_k \subset \mathcal{U}$ ($\mathcal{U}$ for universe) and the sets A_i are *pairwise disjoint*, we have $|A| = |A_1| + \cdots + |A_k|$.

When $A = A_1 \cup \cdots \cup A_k \subset \mathcal{U}$ and the A_i are **not** pairwise disjoint, we must apply the principle of inclusion-exclusion to determine $|A|$:

$$\begin{aligned} |A_1 \cup A_2| &= |A_1| + |A_2| \qquad\qquad - |A_1 \cap A_2| \\ |A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| \\ &\qquad - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| \end{aligned}$$

$$|A_1 \cup \cdots \cup A_m| = \sum |A_i| - \sum |A_i \cap A_j| + \sum \left|A_i \cap A_j \cap A_k\right| \cdots$$

It may be more convenient to apply inclusion/exclusion where the A_i are *forbidden* subsets of $\mathcal{U}$, in which case ____________________.

mmm. . .PIE

The key to using the principle of inclusion-exclusion is determining the right choice of A_i. The A_i and their intersections should be easy to count and easy to characterize.

Notation: $\pi = p_1 p_2 \cdots p_n$ is the one-line notation for a permutation of [n] whose first element is p_1, second element is p_2, etc.

Example. How many permutations $p = p_1 p_2 \cdots p_n$ are there in which at least one of p_1 and p_2 are even?

Solution. Let $\mathcal{U}$ be the set of n-permutations.
Let A_1 be the set of permutations where p_1 is even.
Let A_2 be the set of permutations where p_2 is even.
In words, $A_1 \cap A_2$ is the set of n-permutations ________________

Now calculate: $|A_1| =$ $|A_2| =$
$|A_1 \cap A_2| =$

Applying PIE: So $|A_1 \cup A_2| =$

mmm. . .PIE

Example. Find the number of integers between 1 and 1000 that are **not** divisible by 5, 6, or 8.

Solution. Let $\mathcal{U} = \{n \in \mathbb{Z} \text{ such that } 1 \leq n \leq 1000\}$.
Let $A_1 \subset \mathcal{U}$ be the multiples of 5, $A_2 \subset \mathcal{U}$ be the multiples of 6, and $A_3 \subset \mathcal{U}$ be the multiples of 8. We want $|\mathcal{U}| - |A_1 \cup A_2 \cup A_3|$.

In words, $A_1 \cap A_2$ is the set of integers
$A_1 \cap A_3$ is $A_2 \cap A_3$ is
and $A_1 \cap A_2 \cap A_3$ is the set of integers that are

Now calculate: $|A_1| =$ $|A_2| =$ $|A_3| =$
$|A_1 \cap A_2| =$ $|A_1 \cap A_3| =$ $|A_2 \cap A_3| =$
$|A_1 \cap A_2 \cap A_3| =$

And finally: So $|\mathcal{U}| - |A_1 \cup A_2 \cup A_3| =$

Combinations with Repetitions

Quick review

1. How many ways are there to choose k elements out of the set $\{1 \cdot a_1, 1 \cdot a_2, \cdots, 1 \cdot a_n\}$?

2. How many ways are there to choose k elements out of the set $\{k \cdot a_1, k \cdot a_2, \cdots, k \cdot a_n\}$? $\big($really $\{\infty \cdot a_1, \infty \cdot a_2, \cdots, \infty \cdot a_n\}\big)$

What we would like to calculate is:

In how many ways can we choose k elements out of an arbitrary multiset?

Now, it's as easy as PIE.

Combinations with Repetitions

Example. Determine the number of 10-combinations of the multiset $S = \{3 \cdot a, 4 \cdot b, 5 \cdot c\}$.

Game plan: Let $\mathcal{U}$ be the set of 10-combs of $\{\infty \cdot a, \infty \cdot b, \infty \cdot c\}$. Use PIE to remove the 10-combs that violate the conditions of S

Define A_1 to be 10-combs that include at least ___ a's.
Define A_2 to be 10-combs that include at least ___ b's.
Define A_3 to be 10-combs that include at least ___ c's.

In words, $A_1 \cap A_2$ are those 10-combs that
$A_1 \cap A_3$: $A_2 \cap A_3$:
$A_1 \cap A_2 \cap A_3$

Now calculate: $|\mathcal{U}| =$ $|A_1| =$ $|A_2| = \left(\!\binom{3}{5}\!\right)$ $|A_3| = \left(\!\binom{3}{4}\!\right)$
$|A_1 \cap A_2| = 3$ $|A_1 \cap A_3| = 1$ $|A_2 \cap A_3| = 0$ $|A_1 \cap A_2 \cap A_3| = 0$

And finally: So $|\mathcal{U}| - |A_1 \cup A_2 \cup A_3| =$

Derangements

At a party, 10 partygoers check their hats. They “have a good time”, and are each handed a hat on the way out. In how many ways can the hats be returned so that no one is returned his/her own hat?

This is a derangement of ten objects.

Definition: An n-derangement is an n-permutation $\pi = p_1 p_2 \cdots p_n$ such that $p_1 \neq 1$, $p_2 \neq 2$, $\cdots$, $p_n \neq n$.

Note: A derangement is a permutation without fixed points $\pi(i) = i$.

Notation: We let D_n be the number of all n-derangements.

When you see D_n, think combinatorially: “The number of ways to return n hats to n people so no one gets his/her own hat back”

Calculating the number of derangements

Example. Calculate D_n.

Solution. Let $\mathcal{U}$ be the set of all n-permutations.
Remove bad permutations using PIE.
For all i from 1 to n, define A_i to be n-perms where $p_i = i$.

In words, $A_i \cap A_j$ are n-perms where
$A_i \cap A_j \cap A_k$ are n-perms where
In general, $A_{i_1} \cap \cdots \cap A_{i_k}$ are n-perms with $p_{i_1} = i_1, \cdots, p_{i_k} = i_k$.

Now calculate: $|\mathcal{U}| =$ $|A_1| =$ $|A_2| =$
For all i and j, $|A_i \cap A_j| =$
When intersecting k sets, $|A_{i_1} \cap \cdots \cap A_{i_k}| =$

Recall: $|A_1 \cup \cdots \cup A_n| = \sum |A_i| - \sum |A_i \cap A_j| + \sum \left|A_i \cap A_j \cap A_k\right| \cdots$

Therefore, $D_n = |\mathcal{U}| - |A_1 \cup \cdots \cup A_n| =$

Randomly returning hats

Upon simplification, we see

$$D_n = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n \binom{n}{n} 0!$$
$$= n! - \frac{n!}{1!} + \frac{n!}{2!} - \cdots + (-1)^n \frac{n!}{n!}$$
$$= n! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right]$$

Recall: Taylor series expansion of e^x:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

Plug in $x = -1$ and truncate after n terms to see that

$$e^{-1} \approx \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right]$$

Conclusion: If n people go to a party and the hats are passed back randomly, the probability that no one gets his or her hat back at the party is $D_n/n!$, which is approximately $1/e \approx 37\%$.

Combinatorial proof involving D_n

Recall: The combinatorial interpretation of D_n is: "The number of ways to return n hats to n people so no one gets his/her own hat back"

Example. Prove the following recurrence relation for D_n combinatorially.

$$D_n = (n-1)\big(D_{n-2} + D_{n-1}\big)$$

A formula for Stirling numbers (p. 90)

(Careful: change of notation !!)

Recall: $S(n,k) = \left\{ {n \atop k} \right\}$ is the number of **set partitions** of $[n]$ into exactly k parts, and $k!S(n,k)$ is the number of onto functions $[n]\to[k]$.

Question: What is a formula for $S(n,k)$?

Solution. We will find the number of surjections from $[n]$ to $[k]$.
Use PIE with $\mathcal{U}$ = set of all functions from $[n]$ to $[k]$.
We will remove the "bad" functions where the range is not $[k]$.

Define A_i be the set of functions $f : [n] \to [k]$ where i is not "hit".

In words, $A_{i_1} \cap \cdots \cap A_{i_j}$ are functions where none of i_1 through i_j are elements of the image.

We calculate: $|\mathcal{U}| = k^n$, $|A_i| = (k-1)^n$, $|A_i \cap A_j| = (k-2)^n$
When intersecting j sets, $|A_{i_1} \cap \cdots \cap A_{i_j}| = (k-j)^n$.

Therefore, $k!S(n,k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$; we conclude
$S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n$.

A formula for Bell numbers (p. 166)

(Careful: change of notation !!)

Recall: B_n is the number of partitions of $[n]$ into any number of parts. Manipulate our expression from prev. page to find a formula.

$$B_n = \sum_{k\geq 0} \left\{ {n \atop k} \right\} = \sum_{k\geq 0} \frac{1}{k!} \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} (-1)^{k-j} j^n$$

$$= \sum_{k\geq 0} \sum_{j=0}^{k} \frac{1}{j!(k-j)!} (-1)^{k-j} j^n = \sum_{k\geq 0} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{(k-j)!} \frac{j^n}{j!}$$

$$= \sum_{j\geq 0} \sum_{k\geq j} \frac{(-1)^{k-j}}{(k-j)!} \frac{j^n}{j!} = \sum_{j\geq 0} \frac{j^n}{j!} \sum_{m\geq 0} \frac{(-1)^m}{(m)!} = \sum_{j\geq 0} \frac{j^n}{j!} \frac{1}{e}$$

Theorem 4.3.3. For any $n > 0$, $B_n = \frac{1}{e} \sum_{j\geq 0} \frac{j^n}{j!}$.

For example, $B_5 = \frac{1}{e} \left(\frac{0^5}{0!} + \frac{1^5}{1!} + \frac{2^5}{2!} + \frac{3^5}{3!} + \frac{4^5}{4!} + \frac{5^5}{5!} + \cdots \right) = 52$.