

## Principle of Inclusion-Exclusion

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*Solution.*

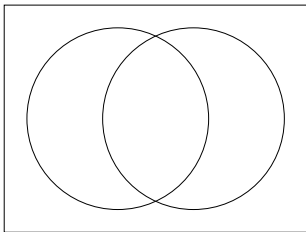
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*Solution.*

Let  $S$  be the set of students who play soccer and  $B$  be the set of students who play basketball.

Then,  $|S \cup B| = |S| + |B|$  \_\_\_\_\_.



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It may be more convenient to apply inclusion/exclusion where the  $A_i$  are *forbidden* subsets of  $\mathcal{U}$ , in which case \_\_\_\_\_.



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**Example.** How many permutations  $p = p_1 p_2 \cdots p_n$  are there in which at least one of  $p_1$  and  $p_2$  are even?

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Let  $A_1$  be the set of permutations where  $p_1$  is even.

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Applying PIE: So  $|A_1 \cup A_2| =$

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Let  $A_1 \subset \mathcal{U}$  be the multiples of 5,  $A_2 \subset \mathcal{U}$  be the multiples of 6, and  $A_3 \subset \mathcal{U}$  be the multiples of 8. We want  $|\mathcal{U}| - |A_1 \cup A_2 \cup A_3|$ .

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And finally: So  $|\mathcal{U}| - |A_1 \cup A_2 \cup A_3| =$

# Combinations with Repetitions

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What we would like to calculate is:

In how many ways can we choose  $k$  elements out of an arbitrary multiset?

Now, it's as easy as PIE.

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Now calculate:  $|\mathcal{U}| = \quad |A_1| = \quad |A_2| = \binom{3}{5} \quad |A_3| = \binom{3}{4}$   
 $|A_1 \cap A_2| = 3 \quad |A_1 \cap A_3| = 1 \quad |A_2 \cap A_3| = 0 \quad |A_1 \cap A_2 \cap A_3| = 0$

And finally: So  $|\mathcal{U}| - |A_1 \cup A_2 \cup A_3| =$

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This is a **derangement** of ten objects.

*Definition:* An  **$n$ -derangement** is an  $n$ -permutation  $\pi = p_1 p_2 \cdots p_n$  such that  $p_1 \neq 1, p_2 \neq 2, \cdots, p_n \neq n$ .

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*Notation:* We let  $D_n$  be the number of all  $n$ -derangements.

When you see  $D_n$ , think combinatorially: “The number of ways to return  $n$  hats to  $n$  people so no one gets his/her own hat back”

## Calculating the number of derangements

**Example.** Calculate  $D_n$ .

**Solution.** Let  $\mathcal{U}$  be the set of all  $n$ -permutations.

Remove bad permutations using PIE.

For all  $i$  from 1 to  $n$ , define  $A_i$  to be  $n$ -perms where  $p_i = i$ .



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Recall:  $|A_1 \cup \dots \cup A_n| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| \dots$

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Therefore,  $D_n = |\mathcal{U}| - |A_1 \cup \dots \cup A_n| =$

## Randomly returning hats

Upon simplification, we see

$$\begin{aligned} D_n &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n \binom{n}{n} 0! \\ &= n! - \frac{n!}{1!} + \frac{n!}{2!} - \cdots + (-1)^n \frac{n!}{n!} \\ &= n! \left[ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right] \end{aligned}$$

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**Recall:** Taylor series expansion of  $e^x$ :

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

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 &= n! \left[ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right]
 \end{aligned}$$

**Recall:** Taylor series expansion of  $e^x$ :

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Plug in  $x = -1$  and truncate after  $n$  terms to see that

$$e^{-1} \approx \left[ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right]$$



## Randomly returning hats

Upon simplification, we see

$$\begin{aligned} D_n &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n \binom{n}{n} 0! \\ &= n! - \frac{n!}{1!} + \frac{n!}{2!} - \cdots + (-1)^n \frac{n!}{n!} \\ &= n! \left[ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right] \end{aligned}$$

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**Conclusion:** If  $n$  people go to a party and the hats are passed back randomly, the probability that no one gets his or her hat back at the party is  $D_n/n!$ , which is approximately  $1/e \approx 37\%$ .

## Combinatorial proof involving $D_n$

**Recall:** The combinatorial interpretation of  $D_n$  is: “The number of ways to return  $n$  hats to  $n$  people so no one gets his/her own hat back”

**Example.** Prove the following recurrence relation for  $D_n$  combinatorially.

$$D_n = (n - 1)(D_{n-2} + D_{n-1})$$

## A formula for Stirling numbers (p. 90)

Recall:  $S(n, k) = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$  is the number of **set partitions** of  $[n]$  into exactly  $k$  parts

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*Solution.* We will find the number of surjections from  $[n]$  to  $[k]$ . Use PIE with  $\mathcal{U}$  = set of **all** functions from  $[n]$  to  $[k]$ .

We will remove the “bad” functions where the range is not  $[k]$ .

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We calculate:  $|\mathcal{U}| = k^n$ ,  $|A_i| = (k - 1)^n$ ,  $|A_i \cap A_j| = (k - 2)^n$   
When intersecting  $j$  sets,  $|A_{i_1} \cap \cdots \cap A_{i_j}| = (k - j)^n$ .

## A formula for Stirling numbers (p. 90)

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When intersecting  $j$  sets,  $|A_{i_1} \cap \cdots \cap A_{i_j}| = (k-j)^n$ .

Therefore,  $k!S(n, k) = \sum_{j=0}^k (-1)^j \binom{k}{j} (k-j)^n$ ; we conclude

$$S(n, k) = \frac{1}{k!} \sum_{j=0}^k (-1)^j \binom{k}{j} (k-j)^n$$

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Therefore,  $k!S(n, k) = \sum_{j=0}^k (-1)^j \binom{k}{j} (k-j)^n$ ; we conclude

$$S(n, k) = \frac{1}{k!} \sum_{j=0}^k (-1)^j \binom{k}{j} (k-j)^n = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^n.$$

## A formula for Bell numbers (p. 166)

(Careful: change of notation !!)

Recall:  $B_n$  is the number of partitions of  $[n]$  into any number of parts. Manipulate our expression from prev. page to find a formula.

$$B_n = \sum_{k \geq 0} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \sum_{k \geq 0} \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} j^n$$

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$$B_n = \sum_{k \geq 0} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \sum_{k \geq 0} \frac{1}{k!} \sum_{j=0}^k \frac{k!}{j!(k-j)!} (-1)^{k-j} j^n$$

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Recall:  $B_n$  is the number of partitions of  $[n]$  into any number of parts. Manipulate our expression from prev. page to find a formula.

$$\begin{aligned} B_n &= \sum_{k \geq 0} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \sum_{k \geq 0} \frac{1}{k!} \sum_{j=0}^k \frac{k!}{j!(k-j)!} (-1)^{k-j} j^n \\ &= \sum_{k \geq 0} \sum_{j=0}^k \frac{1}{j!(k-j)!} (-1)^{k-j} j^n \end{aligned}$$

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Recall:  $B_n$  is the number of partitions of  $[n]$  into any number of parts. Manipulate our expression from prev. page to find a formula.

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 B_n &= \sum_{k \geq 0} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \sum_{k \geq 0} \frac{1}{k!} \sum_{j=0}^k \frac{k!}{j!(k-j)!} (-1)^{k-j} j^n \\
 &= \sum_{k \geq 0} \sum_{j=0}^k \frac{1}{j!(k-j)!} (-1)^{k-j} j^n = \sum_{k \geq 0} \sum_{j=0}^k \frac{(-1)^{k-j}}{(k-j)!} \frac{j^n}{j!}
 \end{aligned}$$



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 &= \sum_{k \geq 0} \sum_{j=0}^k \frac{1}{j!(k-j)!} (-1)^{k-j} j^n = \sum_{k \geq 0} \sum_{j=0}^k \frac{(-1)^{k-j}}{(k-j)!} \frac{j^n}{j!} \\
 &= \sum_{j \geq 0} \sum_{k \geq j} \frac{(-1)^{k-j}}{(k-j)!} \frac{j^n}{j!}
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 &= \sum_{k \geq 0} \sum_{j=0}^k \frac{1}{j!(k-j)!} (-1)^{k-j} j^n = \sum_{k \geq 0} \sum_{j=0}^k \frac{(-1)^{k-j} j^n}{(k-j)! j!} \\
 &= \sum_{j \geq 0} \sum_{k \geq j} \frac{(-1)^{k-j} j^n}{(k-j)! j!} = \sum_{j \geq 0} \frac{j^n}{j!} \sum_{k \geq j} \frac{(-1)^{k-j}}{(k-j)!}
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 &= \sum_{j \geq 0} \sum_{k \geq j} \frac{(-1)^{k-j} j^n}{(k-j)! j!} = \sum_{j \geq 0} \frac{j^n}{j!} \sum_{m \geq 0} \frac{(-1)^m}{(m)!}
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 &= \sum_{k \geq 0} \sum_{j=0}^k \frac{1}{j!(k-j)!} (-1)^{k-j} j^n = \sum_{k \geq 0} \sum_{j=0}^k \frac{(-1)^{k-j} j^n}{(k-j)! j!} \\
 &= \sum_{j \geq 0} \sum_{k \geq j} \frac{(-1)^{k-j} j^n}{(k-j)! j!} = \sum_{j \geq 0} \frac{j^n}{j!} \sum_{m \geq 0} \frac{(-1)^m}{(m)!} = \sum_{j \geq 0} \frac{j^n}{j!} \frac{1}{e}
 \end{aligned}$$

# A formula for Bell numbers (p. 166)

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Recall:  $B_n$  is the number of partitions of  $[n]$  into any number of parts. Manipulate our expression from prev. page to find a formula.

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 B_n &= \sum_{k \geq 0} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \sum_{k \geq 0} \frac{1}{k!} \sum_{j=0}^k \frac{k!}{j!(k-j)!} (-1)^{k-j} j^n \\
 &= \sum_{k \geq 0} \sum_{j=0}^k \frac{1}{j!(k-j)!} (-1)^{k-j} j^n = \sum_{k \geq 0} \sum_{j=0}^k \frac{(-1)^{k-j} j^n}{(k-j)! j!} \\
 &= \sum_{j \geq 0} \sum_{k \geq j} \frac{(-1)^{k-j} j^n}{(k-j)! j!} = \sum_{j \geq 0} \frac{j^n}{j!} \sum_{m \geq 0} \frac{(-1)^m}{(m)!} = \sum_{j \geq 0} \frac{j^n}{j!} \frac{1}{e}
 \end{aligned}$$

**Theorem 4.3.3.** For any  $n > 0$ ,  $B_n = \frac{1}{e} \sum_{j \geq 0} \frac{j^n}{j!}$ .

# A formula for Bell numbers (p. 166)

(Careful: change of notation !!)

Recall:  $B_n$  is the number of partitions of  $[n]$  into any number of parts. Manipulate our expression from prev. page to find a formula.

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 \end{aligned}$$

**Theorem 4.3.3.** For any  $n > 0$ ,  $B_n = \frac{1}{e} \sum_{j \geq 0} \frac{j^n}{j!}$ .

For example,  $B_5 = \frac{1}{e} \left( \frac{0^5}{0!} + \frac{1^5}{1!} + \frac{2^5}{2!} + \frac{3^5}{3!} + \frac{4^5}{4!} + \frac{5^5}{5!} + \dots \right) = 52$ .