

Generating functions

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Definition: For any sequence $\{a_k\}_{k \geq 0} = a_0, a_1, a_2, a_3, \dots$, its **generating function** is the formal power series

$$A(x) = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + \cdots = \sum_{k \geq 0} a_k x^k.$$

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Example. Let f_k be the Fibonacci numbers starting $f_0 = 0$. Then

$$F(x) = \sum_{k \geq 0} f_k x^k = 0 + 1x^1 + 1x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 + \cdots.$$

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We will see that we can simplify this expression greatly. In fact,

$$F(x) = x / (1 - x - x^2).$$

We will call this the **compact form** of the generating function.

Why Generating Functions?

We will use generating functions to:

- ▶ Find an exact formula for the terms of a sequence.
- ▶ Prove identities involving sequences.
- ▶ Understand partitions of integers.
- ▶ Use algebra to solve combinatorial problems.

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Others use generating functions to:

- ▶ Use complex analysis to solve combinatorial problems.
- ▶ Understand the asymptotics of a sequence.
- ▶ Find averages and statistical properties.
- ▶ Understand **something** about a sequence.

Generating function example: Basketball

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Solution. This is a partition of 6 into parts of size at most 3, so 7:

$$\begin{array}{cccc} 3 + 3 & 3 + 2 + 1 & 3 + 1 + 1 + 1 & 2 + 2 + 2 \\ 2 + 2 + 1 + 1 & 2 + 1 + 1 + 1 + 1 & 1 + 1 + 1 + 1 + 1 + 1 & \end{array}$$

What about 98 points?

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Generating functions will help us keep track of the possibilities.

We'll first reanalyze the question of scoring six points and then generalize to larger numbers.

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To start, break down the possible ways of getting six points total into one-point, two-point, and three-point shots.

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How many points could be scored using one-point shots?

$$\begin{array}{cccccccc} 0 \text{ pts} & \text{or} & 1 \text{ pt} & \text{or} & 2 \text{ pts} & \text{or} & 3 \text{ pts} & \text{or} & 4 \text{ pts} & \text{or} & 5 \text{ pts} & \text{or} & 6 \text{ pts} \\ x^0 & + & x^1 & + & x^2 & + & x^3 & + & x^4 & + & x^5 & + & x^6 \end{array}$$

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How many points could be scored using one-point shots?

0 pts or 1 pt or 2 pts or 3 pts or 4 pts or 5 pts or 6 pts

$$x^0 + x^1 + x^2 + x^3 + x^4 + x^5 + x^6$$

How many points could be scored using two-point shots?

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How many points could be scored using two-point shots?

How many points could be scored using three-point shots?

Multiply these algebraic expressions together:

$$1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 7x^6 + 7x^7 + 8x^8 + 8x^9 + \\ 8x^{10} + 7x^{11} + 7x^{12} + 5x^{13} + 4x^{14} + 3x^{15} + 2x^{16} + x^{17} + x^{18}$$

and find the coefficient of the x^6 term.

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Why does this work? A score a from 1-pt, b from 2-pt, c from 3-pt, gives a term in the product of $x^a x^b x^c = x^{a+b+c}$. Collecting like terms makes the coefficient of x^k the number of ways to score k points. (x^{15} ?)

Generating function example: Basketball

In order to take into account **all** the ways to score 98 points, we include more terms in each factor:

$$\text{One-point shots: } 1 + x + x^2 + \cdots + \quad = \underline{\hspace{2cm}}.$$

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Conclusion: The generating function for the number of ways to score any number of points in basketball is

$$b(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)}.$$

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Notation: $[x^k]f(x)$ is the coefficient of x^k in the expansion of the generating function $f(x)$.

Example. $[x^{98}]b(x) = 850$.

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Strategy. Write down a power series for each piece of fruit, multiply them together, and extract the coefficient of x^k .

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Die F : $\{1, 2, 2, 3, 3, 4\}$ and die G : $\{1, 3, 4, 5, 6, 8\}$

Key series

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$$(1+x)^\alpha = \sum_{k \geq 0} \binom{\alpha}{k} x^k$$

$$e^x = \sum_{k \geq 0} \frac{1}{k!} x^k$$

Manipulations on $A(x) = \sum_{k \geq 0} a_k x^k$

Example. Find the coefficient of x^9 in $\frac{1}{(1-4x)^{12}}$.

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Question: Suppose we can factor out a power of x from $A(x)$.
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Example. Find the compact form of $\sum_{k \geq 0} (-3)^{k+2} x^k$.

Answer:

Memories of calculus. . .

With formal power series, we interchange derivatives, integrals, sums.

$$\sum_{k \geq 0} \frac{d}{dx} x^k = \frac{d}{dx} \sum_{k \geq 0} x^k$$
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$$\sum_{k \geq 0} k a_k x^k = \sum_{k \geq 0} p(k) a_k x^k = p\left(x \frac{d}{dx}\right) (A(x))$$

Memories of calculus...

With formal power series, we interchange derivatives, integrals, sums.

$$\sum_{k \geq 0} kx^{k-1} = \sum_{k \geq 0} \frac{d}{dx} x^k = \frac{d}{dx} \sum_{k \geq 0} x^k = \frac{d}{dx} \frac{1}{1-x} = \frac{1}{(1-x)^2}$$

$$\sum_{k \geq 0} \frac{x^{k+1}}{k+1} = \sum_{k \geq 0} \int_0^x x^k dx = \int_0^x \sum_{k \geq 0} x^k dx = \int_0^x \frac{1}{1-x} dx = -\ln |1-x|$$

How these manipulations interact with $A(x) = \sum_{k \geq 0} a_k x^k$:

$$\sum_{k \geq 0} k a_k x^k = \sum_{k \geq 0} p(k) a_k x^k = p\left(x \frac{d}{dx}\right)(A(x))$$

Example. Find $\sum_{k \geq 0} \frac{k^2 + 4k + 5}{k!}$