Solving recurrence relations

Example. Determine a formula for the entries of the sequence $\{a_k\}_{k\geq 0}$ that satisfies $a_0 = 0$ and the recurrence $a_{k+1} = 2a_k + 1$ for $k \geq 0$.

Solution. Use generating functions: define $A(x) = \sum_{k\geq 0} a_k x^k$.

Step 1: Multiply both sides of the recurrence by x^{k+1} and sum over all k:

Step 2: Massage the sums to find copies of $A(x)$.
LHS: Re-index, find missing term; RHS: separate into pieces.

Conversion to functions of $A(x)$:

Solving recurrence relations

Step 3: Solve for the compact form of $A(x)$.

Step 4: Extract the coefficients.

When the degree of the numerator is smaller than the degree of the denominator, we can use *partial fractions!* to determine an expression for $A(x)$ of the form:

$$A(x) = \frac{C_1}{1-2x} + \frac{C_2}{1-x}$$

Solving gives $A(x) = \frac{1}{1-2x} + \frac{-1}{1-x}$; each of which can be expanded:

$$A(x) = \sum_{k\geq 0} 2^k x^k + (-1)\sum_{k\geq 0} 1^k x^k = \sum_{k\geq 0} (2^k - 1)\, x^k$$

Therefore, $a_k = 2^k - 1$.

A closed form for Fibonacci numbers

Example. Solve the recurrence relation $f_k = f_{k-1} + f_{k-2}$ with initial conditions $f_0 = 0$ and $f_1 = 1$.

Solution. Define $F(x) = \sum_{k\geq 0} f_k x^k$ and rewrite the recurrence with indices without subtraction: $f_{k+2} = f_{k+1} + f_k$. Summing over $k \geq 0$,

$$\sum_{k\geq 0} f_{k+2}x^{k+2} = \sum_{k\geq 0}(f_{k+1} + f_k)x^{k+2}$$

$$\sum_{k\geq 0} f_{k+2}x^{k+2} = \sum_{k\geq 0} f_{k+1}x^{k+2} + \sum_{k\geq 0} f_k x^{k+2}$$

$$\sum_{k\geq 0} f_{k+2}x^{k+2} = x\sum_{k\geq 0} f_{k+1}x^{k+1} + x^2\sum_{k\geq 0} f_k x^k$$

$$\sum_{k\geq 2} f_k x^k = x\sum_{k\geq 1} f_k x^k + x^2\sum_{k\geq 0} f_k x^k$$

Therefore,

A closed form for Fibonacci numbers

So the Fibonacci numbers have generating function $x/(1-x-x^2)$.
The roots of $(1-x-x^2) = (1-r_+x)(1-r_-x)$ are $r_\pm = (1 \pm \sqrt{5})/2$.
Using partial fractions,

$$F(x) = \frac{1}{\sqrt{5}}\frac{1}{1-r_+x} - \frac{1}{\sqrt{5}}\frac{1}{1-r_-x}$$

Therefore, $\displaystyle\sum_{k\geq 0} f_k x^k = \sum_{k\geq 0} \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k x^k - \sum_{k\geq 0} \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k x^k$

and we conclude that $\displaystyle f_k = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k$.

As $k \to +\infty$, the second term ____________, so $\displaystyle f_k \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k$

Practicality: $(1+\sqrt{5})/2 \approx 1.61803$ and $1 \text{ mi} \approx 1.609344 \text{ km}$

Multiplying two generating functions (Convolution)

Let $A(x) = \sum_{k \geq 0} a_k x^k$ and $B(x) = \sum_{k \geq 0} b_k x^k$.

Question: What is the coefficient of x^k in $A(x)B(x)$?

When expanding the product $A(x)B(x)$ we multiply terms $a_i x^i$ in A by terms $b_j x^j$ in B. This product contributes to the coefficient of x^k in AB only when ______________________________.

Therefore, $A(x)B(x) = \sum_{k \geq 0} \Big(\underline{\hspace{6em}} \Big) x^k$

Combinatorial interpretation of the convolution:

If a_k counts all "A" objects of "size" k, and
b_k counts all "B" objects of "size" k,

Then $[x^k]\big(A(x)B(x)\big)$ counts all pairs of objects (A, B) with *total* size k.

A Halloween Multiplication

Example. In how many ways can we fill a halloween bag w/30 candies, where for each of 20 BIG candy bars, we can choose at most one, and for each of 40 different small candies, we can choose as many as we like?

Big candy g.f.: $B(x) = (1+x)^{20} = \sum_{k=0}^{\infty} \binom{20}{k} x^k.$

b_k counts (k big candies)

Small candy g.f.: $S(x) = \dfrac{1}{(1-x)^{40}} = \sum_{k=0}^{\infty} \left(\!\binom{40}{k}\!\right) x^k.$

s_k counts (k small candies)

Total g.f.: $B(x)S(x) = \sum_{k=0}^{\infty} \left[\sum_{i=0}^{k} \binom{20}{i} \left(\!\binom{40}{k-i}\!\right) \right] x^k$

Conclusion: $[x^{30}]B(x)S(x) = \sum_{i=0}^{30} \binom{20}{i} \left(\!\binom{40}{30-i}\!\right)$

So, $[x^k]B(x)S(x)$ counts pairs of the form $\vee$ w/k total candies.

(some number of big candies, some number of small candies)

Vandermonde's Identity (p. 117)

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j} \binom{n}{k-j}$$

Combinatorial proof

Generating function proof

Multiplying two generating functions

Example. What is the coefficient of x^7 in $\frac{x^3(1+x)^4}{(1-2x)}$?

Powers of generating functions

A special case of convolution gives the coefficients of powers of a g.f.:

$$\left(A(x)\right)^2 = \sum_{k\geq 0}\left(\sum_{i=0}^{k} a_i a_{k-i}\right)x^k = \sum_{k\geq 0}\left(\sum_{i_1+i_2=k} a_{i_1}a_{i_2}\right)x^k.$$

Similarly, $\left(A(x)\right)^n = \sum_{k\geq 0}\left(\sum_{i_1+i_2+\cdots+i_n=k} a_{i_1}a_{i_2}\cdots a_{i_n}\right)x^k.$

$[x^k](A(x))^n$ counts *sequences* of objects $(A_1, A_2, \ldots, A_n)$, all of type A, with a total size over all objects of k.