

Combinatorial statistics

Given a set of combinatorial objects \mathcal{A} , a **combinatorial statistic** is an integer given to every element of the set.

In other words, it is a function $\mathcal{A} \rightarrow \mathbb{Z}_{\geq 0}$.

Example. Let \mathcal{S} be the set of subsets of $\{1, 2, 3\}$.

The cardinality of a set is a combinatorial statistic on \mathcal{S} .

$$\begin{array}{cccc} |\emptyset| = 0 & |\{1\}| = 1 & |\{2\}| = 1 & |\{3\}| = 1 \\ |\{1, 2\}| = 2 & |\{1, 3\}| = 2 & |\{2, 3\}| = 2 & |\{1, 2, 3\}| = 3 \end{array}$$

Combinatorial statistics provide a *refinement* of counting.

less information

more information

counting

8

statistics

0	1	2	3
1	3	3	1

**complete
enumeration**

\emptyset $\{1\}$ $\{2\}$ $\{3\}$
 $\{1, 2\}$ $\{1, 3\}$ $\{2, 3\}$ $\{1, 2, 3\}$

Statistics and Permutations

Questions involving combinatorial statistics:

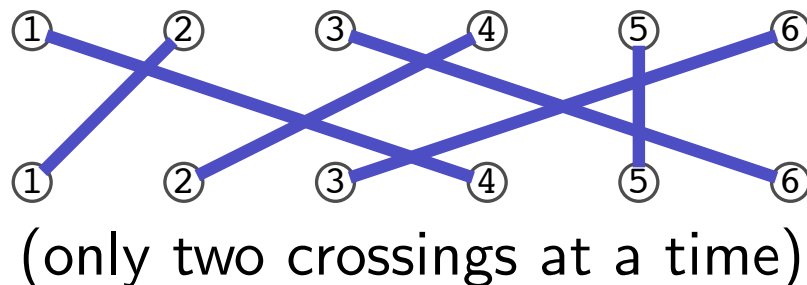
- ▶ What is the *distribution* of the statistics?
- ▶ What is the *average size* of an object in the set?
- ▶ Which statistics have the same distribution?
 - ▶ Insight into their structure.
 - ▶ Provides non-trivial bijections in the set?

A especially rich playground involves *permutation statistics*.

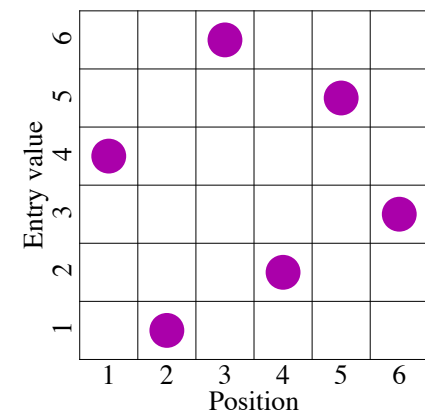
Representations of permutations

One-line notation: $\pi = 416253$ Cycle notation: $\pi = (142)(36)(5)$

String diagram:

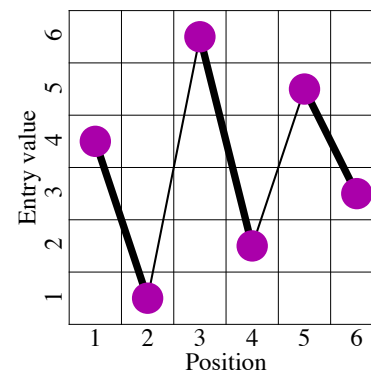


Matrix-like
diagram:



Descent statistic

Definition: Let $\pi = \pi_1\pi_2 \cdots \pi_n$ be a permutation. A **descent** is a position i such that $\pi_i > \pi_{i+1}$. Define $\text{des}(\pi)$ to be the **number of descents** in π .



Example. When $\pi = 416253$, $\text{des}(\pi) = 3$ since $4 \searrow 1$, $6 \searrow 2$, $5 \searrow 3$.

Question: How many n -permutations have d descents?

$\text{des}(12) = 0$ $\text{des}(123) = \underline{\quad}$ $\text{des}(213) = \underline{\quad}$ $\text{des}(312) = \underline{\quad}$
 $\text{des}(21) = 1$ $\text{des}(132) = \underline{\quad}$ $\text{des}(231) = \underline{\quad}$ $\text{des}(321) = \underline{\quad}$

$n \setminus d$	0	1	2	3	4
1	1				
2	1	1			
3	1	4	1		
4	1	11	11	1	
5	1	26	66	26	1

What are the possible values for $\text{des}(\pi)$?

Note the symmetry. If π has d descents, its reverse $\hat{\pi}$ has _____ descents.

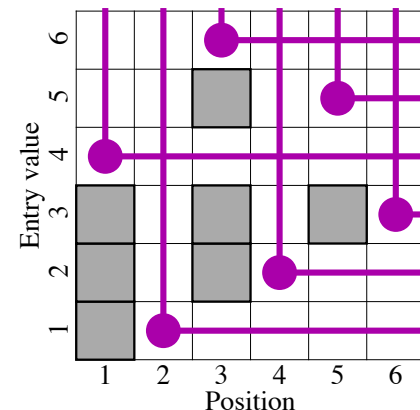
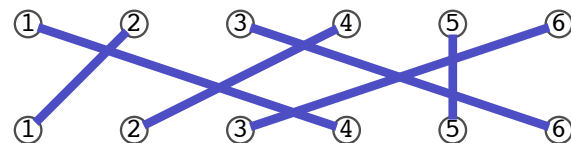
These are the **Eulerian numbers**.

Inversion statistic

Definition: Let $\pi = \pi_1\pi_2 \cdots \pi_n$ be a permutation. An **inversion** is a pair $i < j$ such that $\pi_i > \pi_j$.

Define $\text{inv}(\pi)$ as the **number of inversions** in π .

Example. When $\pi = 416253$, $\text{inv}(\pi) = 7$ since $4 > 1, 4 > 2, 4 > 3, 6 > 2, 6 > 5, 6 > 3, 5 > 3$. In a string diagram $\text{inv}(\pi) =$ number of crossings. In a matrix diagram $\text{inv}(\pi)$, draw **Rothe diagram**:



$$\begin{array}{cccc} \text{inv}(12) = 0 & \text{inv}(123) = \underline{\quad} & \text{inv}(213) = \underline{\quad} & \text{inv}(312) = \underline{\quad} \\ \text{inv}(21) = 1 & \text{inv}(132) = \underline{\quad} & \text{inv}(231) = \underline{\quad} & \text{inv}(321) = \underline{\quad} \end{array}$$

$n \setminus i$	0	1	2	3	4	5	6
1	1						
2	1	1					
3	1	2	2	1			
4	1	3	5	6	5	3	1

What are the possible values for $\text{inv}(\pi)$?

The inversion number is a good way to count how “far away” a permutation is from the identity.

Major index

Definition: Let $\pi = \pi_1\pi_2 \cdots \pi_n$ be a permutation.

Define $\text{maj}(\pi)$, the **major index** of π , to be sum of the descents of π .

[Named after Major Percy MacMahon. (British army, early 1900's)]

Example. When $\pi = 416253$, $\text{maj}(\pi) = 9$ since the descents of π are in positions 1, 3, and 5.

$$\begin{array}{llll} \text{maj}(12) = 0 & \text{maj}(123) = \underline{\quad} & \text{maj}(213) = \underline{\quad} & \text{maj}(312) = \underline{\quad} \\ \text{maj}(21) = 1 & \text{maj}(132) = \underline{\quad} & \text{maj}(231) = \underline{\quad} & \text{maj}(321) = \underline{\quad} \end{array}$$

$n \setminus m$	0	1	2	3	4	5	6
1	1						
2	1	1					
3	1	2	2	1			
4	1	3	5	6	5	3	1

What are the possible values for $\text{maj}(\pi)$?

The distribution of $\text{maj}(\pi)$
IS THE SAME AS
the distribution of $\text{inv}(\pi)$!

A statistic that has the same distribution as inv is called **Mahonian**.

q-analogs

Definition: A **q-analog** of a number c is an expression $f(q)$ such that $\lim_{q \rightarrow 1} f(q) = c$.

Example. $\frac{1 - q^n}{1 - q} = (1 + q + q^2 + \cdots + q^{n-2} + q^{n-1})$ is a q-analog of n because $\lim_{q \rightarrow 1} \frac{1 - q^n}{1 - q} = n$.

We write $[n]_q = \frac{1 - q^n}{1 - q}$.

q-analogs work hand in hand with combinatorial statistics.

If stat is a combinatorial statistic on a set S ($\text{stat} : S \mapsto \mathbb{N}$), then $\sum_{s \in S} q^{\text{stat}(s)}$ is a q-analog of $|S|$ because

$$\lim_{q \rightarrow 1} \sum_{s \in S} q^{\text{stat}(s)} = \sum_{s \in S} 1^{\text{stat}(s)} = \sum_{s \in S} 1 = |S|$$

Inversion statistics

Question: What is the generating function $\sum_{\pi \in S_n} q^{\text{inv}(\pi)}$?

n	$\sum_{\pi \in S_n} q^{\text{inv}(\pi)}$	
1	$1q^0$	$= 1$
2	$1q^0 + 1q^1$	$= (1 + q)$
3	$1q^0 + 2q^1 + 2q^2 + 1q^3$	$= (1 + q + q^2)(1 + q)$
4	$1q^0 + 3q^1 + 5q^2 + 6q^3 + 5q^4 + 3q^5 + 1q^6$	$=$

Conjecture: $\sum_{\pi \in S_n} q^{\text{inv}(\pi)} = [n]_q \cdots [1]_q =: [n]_q!$, the **q-factorial**.

Claim: This equation makes sense when $q = 1$.

Inversion Statistics

Theorem: $\sum_{\pi \in S_n} q^{\text{inv}(\pi)} = [n]_q!$

Proof. There exists a bijection

$$\left\{ \begin{array}{c} \text{permutations} \\ \pi \in S_n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{lists } (a_1, \dots, a_n) \\ \text{where } 0 \leq a_i \leq n - i \end{array} \right\}.$$

Given a permutation π , create its **inversion table**. Define a_i to be the number of entries j to the left of i that are smaller than i .

Then $\text{inv}(\pi) = a_1 + a_2 + \dots + a_n$.

Example. The inversion table of $\pi = 43152$ is $(3, 2, 0, 1, 0)$.

$$\begin{aligned} \sum_{\pi \in S_n} q^{\text{inv}(\pi)} &= \sum_{a_1=0}^{n-1} \sum_{a_2=0}^{n-2} \dots \sum_{a_n=0}^0 q^{a_1+a_2+\dots+a_n} \\ &= \left(\sum_{a_1=0}^{n-1} q^{a_1} \right) \left(\sum_{a_2=0}^{n-2} q^{a_2} \right) \dots \left(\sum_{a_n=0}^0 q^{a_n} \right) \\ &= [n]_q [n-1]_q \dots [1]_q = [n]_q! \end{aligned}$$

Notes

We said that inv and maj are equidistributed. Two possible proofs:

- ▶ Find a bijection $f : S_n \rightarrow S_n$ such that $\text{maj}(\pi) = \text{inv}(f(\pi))$.
- ▶ Or prove $\sum_{\pi \in S_n} q^{\text{inv}(\pi)} = \sum_{\pi \in S_n} q^{\text{maj}(\pi)}$.

With a q -analog of factorials, we can define a q -analog of binomial coefficients. Define

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

These *polynomials* are called the **q -binomial coefficients** or **Gaussian polynomials**.

- ▶ $\lim_{q \rightarrow 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}$.
- ▶ They are indeed polynomials.
- ▶ **Example.** $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$

Combinatorial interpretations of q -binomial coefficients!

Combinatorial interpretations of q -binomial coefficients

Consider set $S_{k,n-k}$ of permutations of the multiset $\{1^k, 2^{n-k}\}$.
 Define $\text{inv}(\pi) = |\{i < j: \pi(i) > \pi(j)\}|$.

Example. $\pi = 1122121122$ is a permutation of $\{1^5, 2^5\}$.
 Then $\text{inv}(\pi) = 0 + 0 + 3 + 3 + 0 + 2 + 0 + 0 + 0 + 0 = 8$.

Then $\sum_{\pi \in S_{k,n-k}} q^{\text{inv}(\pi)} = \begin{bmatrix} n \\ k \end{bmatrix}_q$. (Note $|S_{k,n-k}| = \binom{n}{k}$.)

This is a refinement of these permutations in terms of inversions.

Consider the set \mathcal{P} of lattice paths from $(0,0)$ to (a,b) .




Let $\text{area}(P)$ be the area above a path P .

Then $\sum_{P \in \mathcal{P}} q^{\text{area}(P)} = \begin{bmatrix} a+b \\ a \end{bmatrix}_q$. (Note $|\mathcal{P}| = \binom{a+b}{a}$.)

This can also be used to give a q -analog of the Catalan numbers.

There's always more to learn!!!

References :

-  [Miklós Bóna](#). Combinatorics of Permutations, CRC, 2004.
-  [T. Kyle Petersen](#). Two-sided Eulerian numbers via balls in boxes.
<http://arxiv.org/abs/1209.6273>
-  [The Combinatorial Statistic Finder](#). <http://findstat.org/>