Combinatorics of Core Partitions

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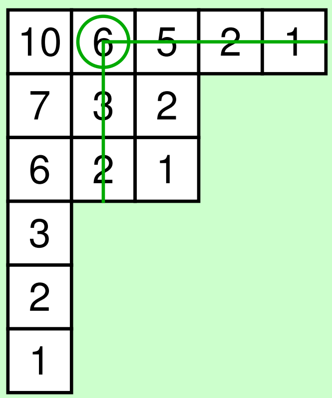
Partitions

The **Young diagram** of $\lambda = (\lambda_1, \ldots, \lambda_k)$ has λ_i boxes in row i.

The **hook length** of a box = # boxes below + # boxes to right + box

λ is an ***a*-core** if no boxes have hook length a.



4-Core Partition

$\lambda = (5, 3, 3, 1, 1, 1)$

Simultaneous

$(4, 7)$-core partition

- There are infinitely many a-core partitions. $(a \geq 2)$

Of interest: Partitions that are **both** a-core **and** b-core. $(a, b) = 1$

- (Anderson, 2002): # (a, b)-core partitions equals $\frac{1}{a+b}\binom{a+b}{a}$.

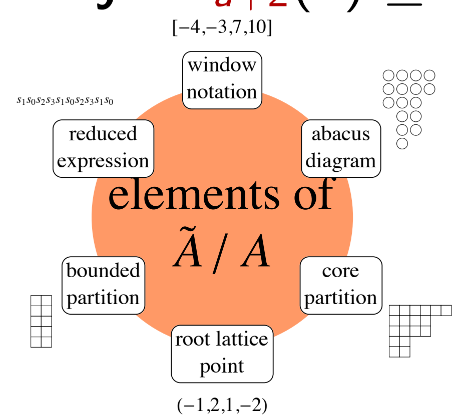
Core partitions in the literature

- **Representation Theory:** (origin)
  - Nakayama conjecture, proved by Brauer & Robinson 1947 says a-cores label a-blocks of irreducible modular representations for S_n.
- **Number Theory:**
  - Let $c_a(n) = \#$ of a-core partitions of n.
  - In 1976, Olsson proved $\displaystyle\sum_{n\geq 0} c_a(n)x^n = \prod_{n\geq 1} \frac{(1-x^{na})^a}{1-x^n}$

  Numerical properties of $c_a(n)$?

  - 1996: Granville & Ono proved **positivity:** $c_a(n) > 0$ $(a \geq 4)$.
  - 1999: Stanton conjectured **monotonicity:** $c_{a+1}(n) \geq c_a(n)$
  - 2012: R. Nath & I conjectured **monotonicity:** $sc_{a+2}(n) \geq sc_a(n)$
- **Modular forms:** g.f. related to Dedekind's η-fcn, a m.f. of wt. $1/2$.
- **Group Theory:** By Lascoux 2001, a-cores $\longleftrightarrow$ coset reps in $\widetilde{S}_a/S_a$

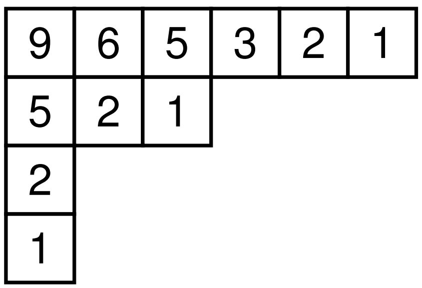
  Group actions on combinatorial objects!!!!

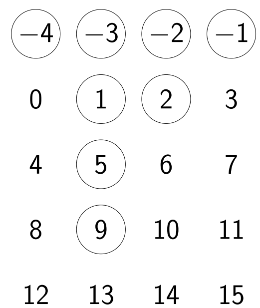


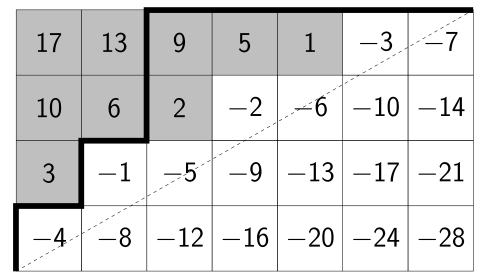
Anderson's bijection and the formula

Building on James's abacus diagrams, Anderson found a bijection:

$$\left\{\begin{matrix}\text{simultaneous}\\ (a,b)\text{-cores}\end{matrix}\right\} \overset{James}{\longleftrightarrow} \left\{\begin{matrix}(a,b)\text{-flush}\\ \text{abaci}\end{matrix}\right\} \overset{And.}{\longleftrightarrow} \left\{\begin{matrix}(a,b)\text{-Dyck paths}\\ (0,0)\to(b,a)\\ \text{above } y=\frac{a}{b}x\end{matrix}\right\}$$







Proof that the number of (a,b)-Dyck paths is $\frac{1}{a+b}\binom{a+b}{a}$:

- Path rotation gives an equivalence relation on the set of all lattice paths from $(0,0) \to (b,a)$.
- There are $\binom{a+b}{a}$ such paths and the equivalence classes have $a+b$ elements each.

Familiar numbers

a	1	2	3	4	5	6	n
# of $(a, a+1)$-cores:	1	2	5	14	42	132	

Specialize Anderson's result: # $(t, t+1)$-cores $\frac{1}{2t+1}\binom{2t+1}{t} = \frac{1}{t+1}\binom{2t}{t}$

Question: Is there a simple statistic on simultaneous core partitions that gives us a q-analog of the Catalan numbers?

$$\sum_{\substack{\lambda \text{ is a} \\ (t,t+1)\text{-core}}} q^{\text{stat}(\lambda)} = \frac{1}{[t+1]_q} \begin{bmatrix} 2t \\ t \end{bmatrix}_q$$

Answer: Yes. We will create an analog of the **major statistic**.

The major statistic

For a permutation $\pi = \pi_1\pi_2\cdots\pi_n$, the **major statistic** $\mathrm{maj}(\pi)$ is the sum of the positions of the descents of π:

$$\mathrm{maj}(\pi) = \sum_{i:\pi_{i-1}>\pi_i} i.$$

For a $(t, t+1)$-core λ, create the sequence $b = (b_0, \ldots, b_{t-1})$, where $b_i = \#\ 1^{\text{st}}$ col. boxes with hook length $\equiv i \bmod t$.

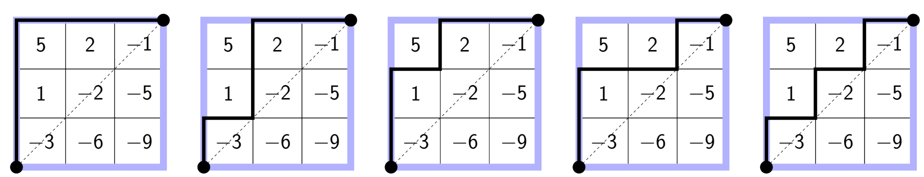
Define $$\mathrm{maj}(\lambda) = \sum_{i\,:\,b_{i-1}\geq b_i} (2i - b_i).$$

Theorem. (AHJ '13)

$$\sum_{\substack{\lambda \text{ is a} \\ (t,t+1)\text{-core}}} q^{\mathrm{maj}(\lambda)} = \frac{1}{[t+1]_q}\begin{bmatrix}2t\\t\end{bmatrix}_q$$

See: maj defined as a sum over descents in a sequence.

Why? Major index on Dyck paths!



Add positions of valleys: $\frac{1}{[4]_q}\begin{bmatrix}6\\3\end{bmatrix}_q = q^0 + q^2 + q^3 + q^4 + q^{2+4}$

Reflection Groups

The combinatorics of **groups**:

- Made up of a set of elements $W = \{w_1, w_2, \ldots\}$.
- Multiplication of two elements $w_1 w_2$ stays in the group.
  - ALTHOUGH, it is **not** the case that $w_1 w_2 = w_2 w_1$.
- There is an identity element (id) & Every element has an inverse.
- Think: (Non-zero real numbers) or (invertible $n \times n$ matrices.)

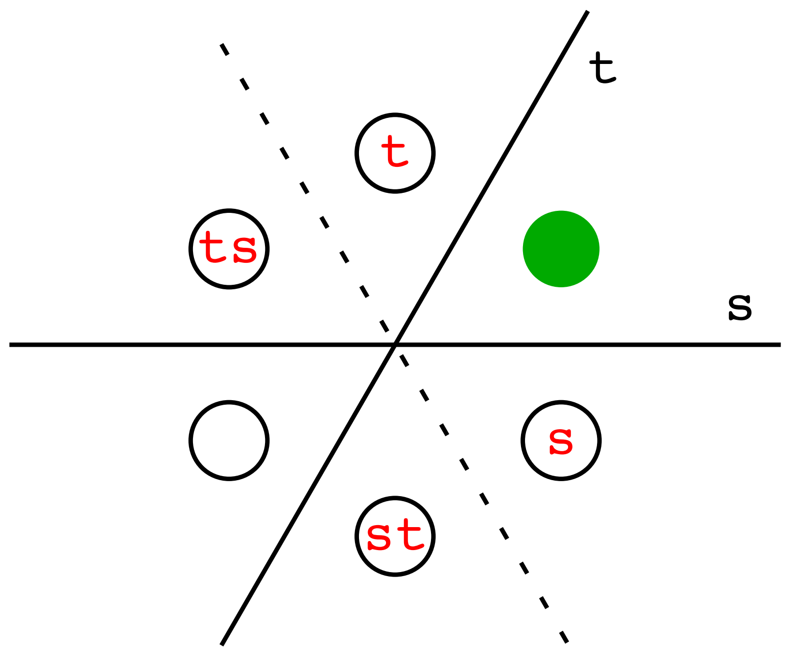
We will talk about **reflection groups**. (With nice pictures)

- W is **generated** by a set of generators $S = \{s_1, s_2, \ldots, s_k\}$.
  - Every $w \in W$ can be written as a product of generators.
- Along with a set of **relations**.
  - These are rules to convert between expressions.
  - $s_i^2 = \text{id}$. —and— $(s_i s_j)^{\text{power}} = \text{id}$.

For example, $w = s_3 s_2 s_1 s_1 s_2 s_4 = s_3 s_2 \text{id} s_2 s_4 = s_3 \text{id} s_4 = s_3 s_4$

Reflection Groups

- The action of multiplying (on the left) by a generator s corresponds to a reflection across a hyperplane H_s. $(s_i^2 = \text{id})$

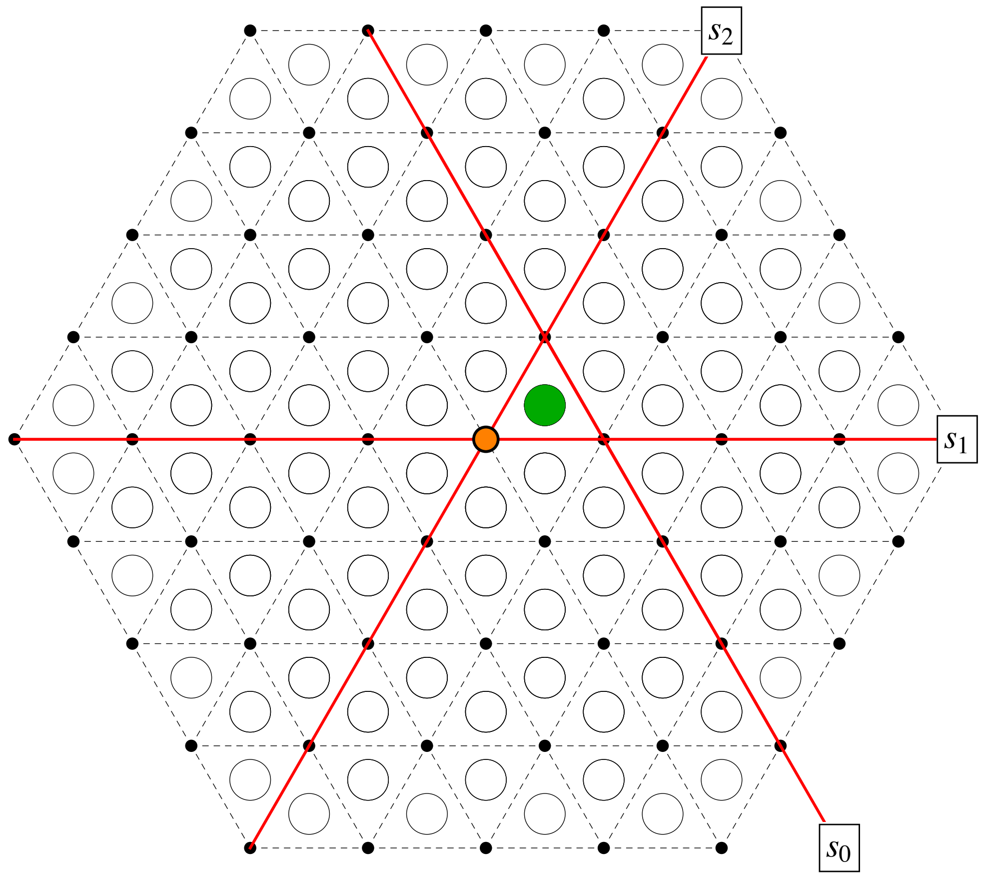


- When the angle between H_s and H_t is $\frac{\pi}{3}$, relation is $(st)^3 = \text{id}$.
- The group depends on the placement of the hyperplanes. $|S| = 6$.

Infinite Reflection Groups

An infinite reflection group: the **affine permutations** $\widetilde{S}_n$.

- Add a new generator s_0 and a new affine hyperplane H_0.



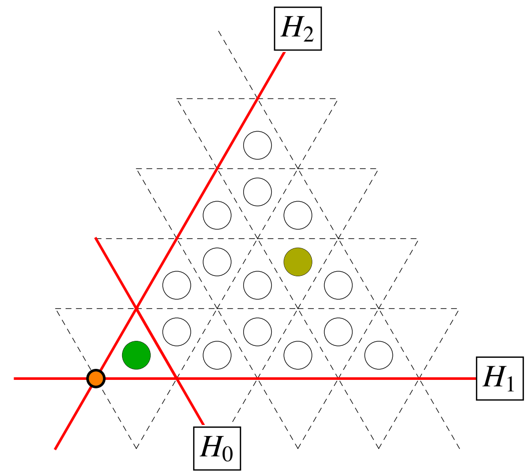
Elements generated by $\{s_0, s_1, s_2\}$ correspond to **alcoves** here.

Combinatorics of affine permutations

Many ways to reference elements in $\widetilde{S}_n$.

- **Geometry.** Point to the alcove.
- **Alcove coordinates.** Keep track of how many hyperplanes of each type you have crossed to get to your alcove.
- **Word.** Write the element as a (short) product of generators.
- **Permutation.** Similar to writing finite permutations as 312.
- **Abacus diagram.** Columns of numbers.
- **Core partition.** Hook length condition.
- **Bounded partition.** Part size bounded.
- **Others!** Lattice path, order ideal, etc.

They all play nicely with each other.



Coordinates:

3	1
1	

Word: $s_0 s_1 s_2 s_1 s_0$

Permutation:

$$(-3, 2, 7)$$

Action of generators on the core partition

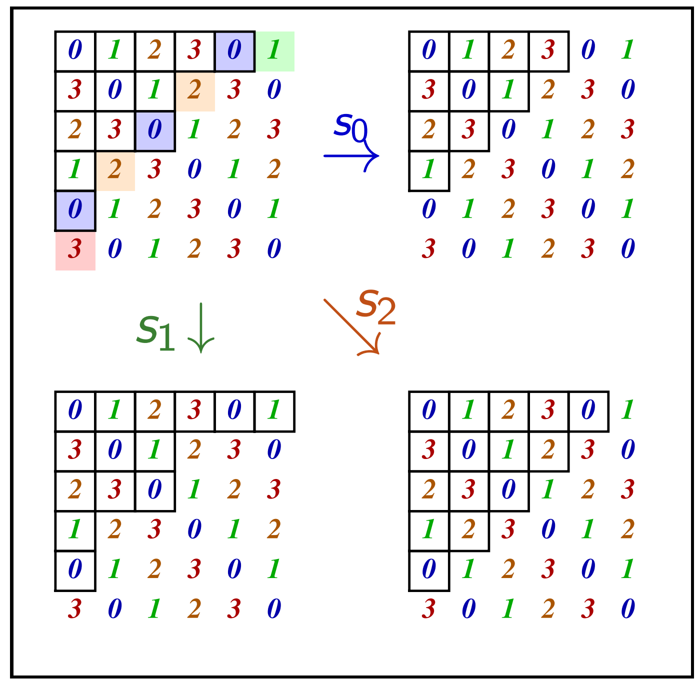
- Label the boxes of λ with residues.
- s_i acts by adding or removing boxes with residue i.

0	1	2	3	0	1
3	0	1	2	3	0
2	3	0	1	2	3
1	2	3	0	1	2
0	1	2	3	0	1
3	0	1	2	3	0

Example. $\lambda = (5,3,3,1,1)$ is a 4-core.

- has removable 0 boxes
- has addable 1, 2, 3 boxes.

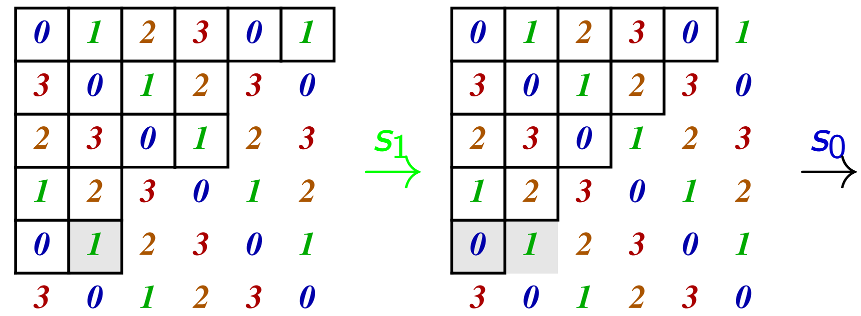
Idea: We can use this to figure out a *word* for w.

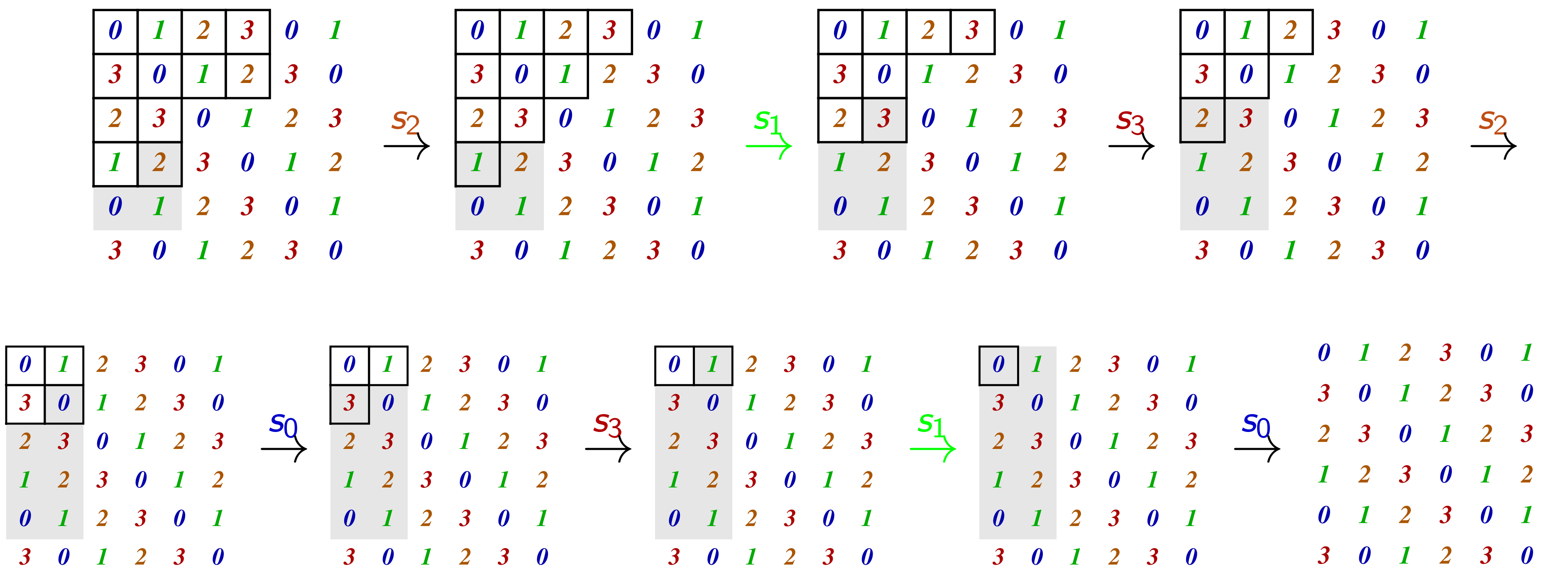


Finding a word for an affine permutation.

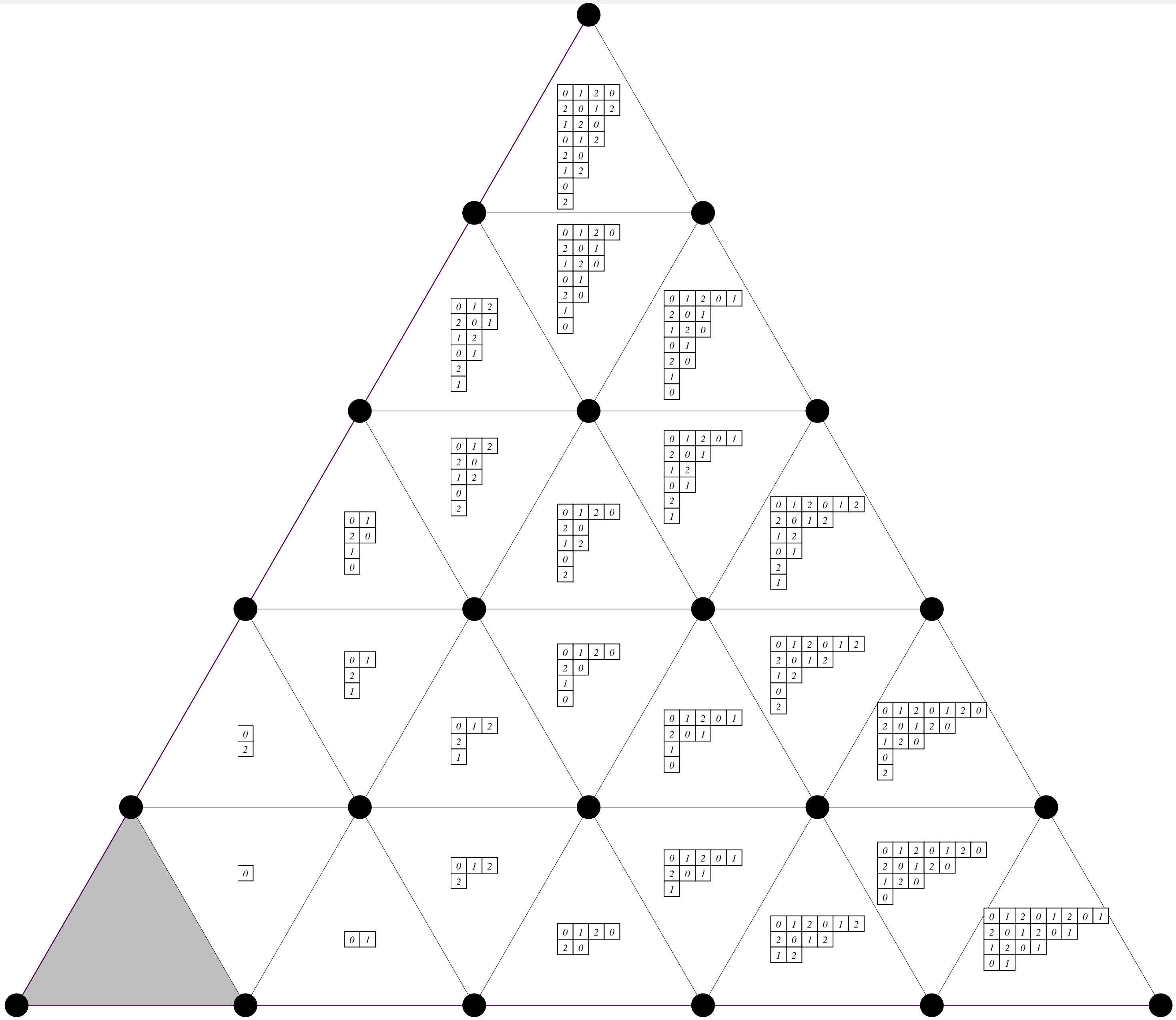
Example: The word in S_4 corresponding to $\lambda = (6, 4, 4, 2, 2)$:

$s_1 s_0 s_2 s_1 s_3 s_2 s_0 s_3 s_1 s_0$





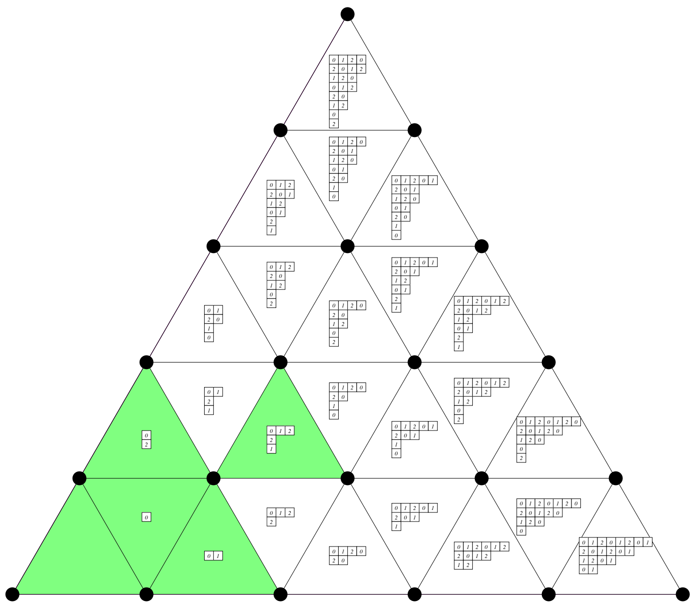
The bijection between cores and alcoves



Simultaneous core partitions

Fishel–Vazirani proved an alcove interpretation of $n/(mn+1)$-cores.

How many partitions are both 3-cores and 2-cores? $C_2 = 2$.



How many partitions are both 3-cores and 4-cores? $C_3 = 5$.

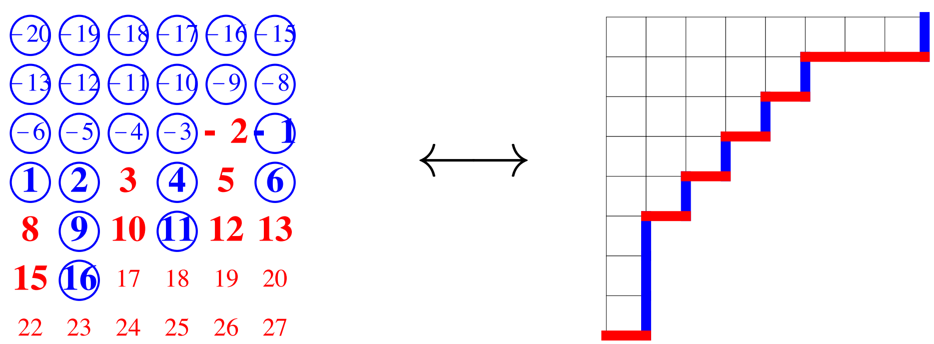
From Anderson's formula $\frac{1}{a+b}\binom{a+b}{a}$,

The number of $(3,7)$-cores is $\frac{1}{10}\binom{10}{3} = \frac{1}{10}\frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1} = 12$.

Research Questions

★ Can we extend combinatorial interps to other reflection groups?

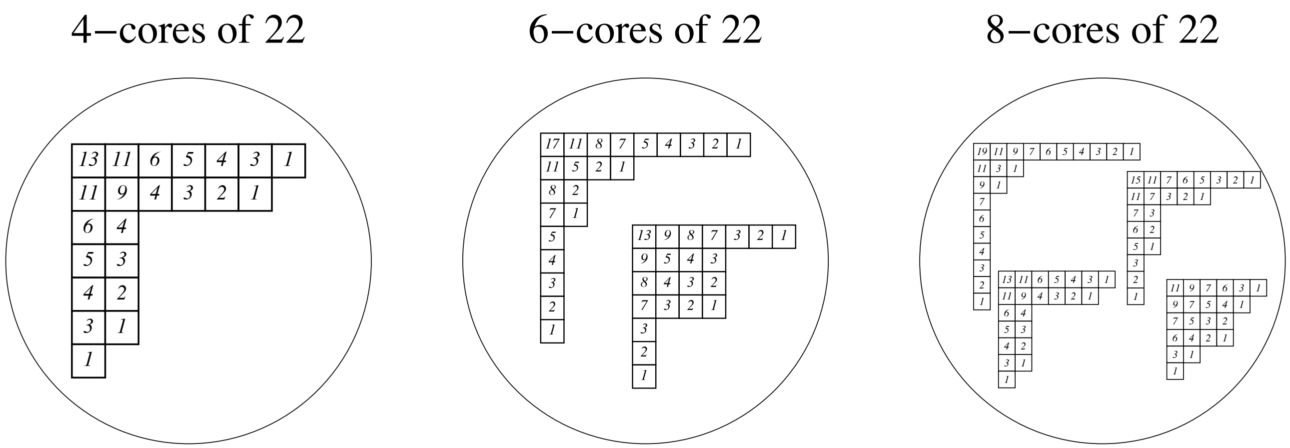
- Yes! Involves self-conjugate partitions.
- Article (28 pp) published in *Journal of Algebra.* (2012)
  Sets up the theory.
- Article (16 pp) published in *European Journal of Comb.*
  Applies the theory.
- Joint with Brant Jones, JMU, Drew Armstrong, Miami.



Research Questions

★ What numerical properties do self-conjugate core partitions have?

- There are more (s.c. $t+2$-cores of n) than (s.c. t-cores of n).
- Article (17 pp) published in *Journal of Number Theory.* (2013)
- Joint with Rishi Nath, York College, CUNY.



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★ Properties of simultaneous core partitions. (Formula: $\frac{1}{s+t}\binom{s+t}{s}$)

- **Question.** What is the average size of an (s,t)-core partition?
- *Progress:* Answer: $(s+t+1)(s-1)(t-1)/24$. Proof?
- **Question:** Is there a core statistic for a q-analog of $\frac{1}{s+t}\binom{s+t}{s}$?
- *Progress:* m-Catalan number C_3 through $(3, 3m+1)$-cores.
- **Question:** Why is the zeta map a bijection?
- *Progress:* An inverse in certain cases.
- Joint with Tom Denton and Cesar Ceballos.