Combinatorics of Core Partitions

Christopher R. H. Hanusa

Queens College, CUNY

Joint work with Brant Jones, James Madison University
Drew Armstrong, University of Miami
Rishi Nath, York College, CUNY
Tom Denton, Google
Cesar Ceballos, York University, Toronto

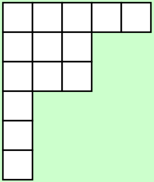
http://qcpages.qc.edu/~chanusa/research

Partitions

The **Young diagram** of $\lambda = (\lambda_1, \ldots, \lambda_k)$ has λ_i boxes in row i.

The **hook length** of a box = # boxes below + # boxes to right + box

λ is an ***a*-core** if no boxes have hook length a.



Partition

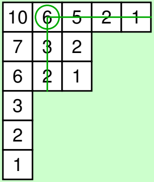
$\lambda = (5, 3, 3, 1, 1, 1)$

Partitions

The **Young diagram** of $\lambda = (\lambda_1, \dots, \lambda_k)$ has λ_i boxes in row i.

The **hook length** of a box = # boxes below + # boxes to right + box

λ is an ***a*-core** if no boxes have hook length a.



Partition
$\lambda = (5, 3, 3, 1, 1, 1)$

Partitions

The **Young diagram** of $\lambda = (\lambda_1, \ldots, \lambda_k)$ has λ_i boxes in row i.

The **hook length** of a box = # boxes below + # boxes to right + box

λ is an ***a*-core** if no boxes have hook length a.

10	6	5	2	1
7	3	2		
6	2	1		
3				
2				
1				

4-Core Partition

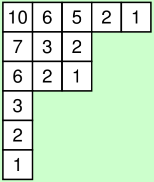
$\lambda = (5, 3, 3, 1, 1, 1)$

Partitions

The **Young diagram** of $\lambda = (\lambda_1, \ldots, \lambda_k)$ has λ_i boxes in row i.

The **hook length** of a box = # boxes below + # boxes to right + box

λ is an ***a*-core** if no boxes have hook length a.



4-Core Partition

$\lambda = (5, 3, 3, 1, 1, 1)$

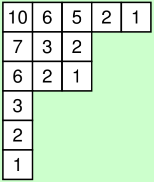
- There are infinitely many a-core partitions. ($a \geq 2$)

Partitions

The **Young diagram** of $\lambda = (\lambda_1, \ldots, \lambda_k)$ has λ_i boxes in row i.

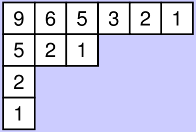
The **hook length** of a box = # boxes below + # boxes to right + box

λ is an ***a*-core** if no boxes have hook length a.



4-Core Partition

$\lambda = (5, 3, 3, 1, 1, 1)$



Simultaneous

$(4, 7)$-core partition

- There are infinitely many a-core partitions. $(a \geq 2)$

Of interest: Partitions that are **both** a-core **and** b-core. $(a, b) = 1$

Partitions

The **Young diagram** of $\lambda = (\lambda_1, \ldots, \lambda_k)$ has λ_i boxes in row i.

The **hook length** of a box = # boxes below + # boxes to right + box

λ is an ***a*-core** if no boxes have hook length a.

10	6	5	2	1
7	3	2		
6	2	1		
3				
2				
1				

4-Core Partition

$\lambda = (5, 3, 3, 1, 1, 1)$

9	6	5	3	2	1
5	2	1			
2					
1					

Simultaneous

$(4, 7)$-core partition

- There are infinitely many a-core partitions. $(a \geq 2)$

Of interest: Partitions that are **both** a-core **and** b-core. $(a, b) = 1$

- (Anderson, 2002): # (a, b)-core partitions equals $\frac{1}{a+b}\binom{a+b}{a}$.

Core partitions in the literature

- **Representation Theory:** (origin)
  - Nakayama conjecture, proved by Brauer & Robinson 1947 says a-cores label a-blocks of irreducible modular representations for S_n.

Core partitions in the literature

- **Representation Theory:** (origin)
  - Nakayama conjecture, proved by Brauer & Robinson 1947 says a-cores label a-blocks of irreducible modular representations for S_n.
- **Number Theory:**
  - Let $c_a(n) = \#$ of a-core partitions of n.
  - In 1976, Olsson proved $\sum_{n\geq 0} c_a(n)x^n = \prod_{n\geq 1} \frac{(1-x^{na})^a}{1-x^n}$

Core partitions in the literature

- **Representation Theory:** (origin)
  - Nakayama conjecture, proved by Brauer & Robinson 1947 says a-cores label a-blocks of irreducible modular representations for S_n.
- **Number Theory:**
  - Let $c_a(n) = \#$ of a-core partitions of n.
  - In 1976, Olsson proved $\sum_{n\geq 0} c_a(n)x^n = \prod_{n\geq 1} \frac{(1-x^{na})^a}{1-x^n}$

Numerical properties of $c_a(n)$?

- 1996: Granville & Ono proved **positivity:** $c_a(n) > 0$ $(a \geq 4)$.
- 1999: Stanton conjectured **monotonicity:** $c_{a+1}(n) \geq c_a(n)$
- 2012: R. Nath & I conjectured **monotonicity:** $sc_{a+2}(n) \geq sc_a(n)$

Core partitions in the literature

- **Representation Theory:** (origin)
  - Nakayama conjecture, proved by Brauer & Robinson 1947 says a-cores label a-blocks of irreducible modular representations for S_n.
- **Number Theory:**
  - Let $c_a(n) = \#$ of a-core partitions of n.
  - In 1976, Olsson proved $\sum_{n\geq 0} c_a(n)x^n = \prod_{n\geq 1} \frac{(1-x^{na})^a}{1-x^n}$

  Numerical properties of $c_a(n)$?

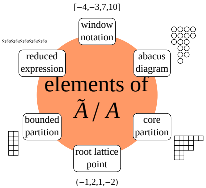
  - 1996: Granville & Ono proved **positivity:** $c_a(n) > 0$ $(a \geq 4)$.
  - 1999: Stanton conjectured **monotonicity:** $c_{a+1}(n) \geq c_a(n)$
  - 2012: R. Nath & I conjectured **monotonicity:** $sc_{a+2}(n) \geq sc_a(n)$
- **Modular forms:** g.f. related to Dedekind's η-fcn, a m.f. of wt. $1/2$.

Core partitions in the literature

- **Representation Theory:** (origin)
  - Nakayama conjecture, proved by Brauer & Robinson 1947 says a-cores label a-blocks of irreducible modular representations for S_n.
- **Number Theory:**
  - Let $c_a(n) = \#$ of a-core partitions of n.
  - In 1976, Olsson proved $\sum_{n\geq 0} c_a(n)x^n = \prod_{n\geq 1} \frac{(1-x^{na})^a}{1-x^n}$

  Numerical properties of $c_a(n)$?

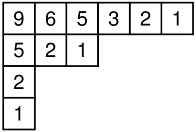
  - 1996: Granville & Ono proved **positivity:** $c_a(n) > 0$ $(a \geq 4)$.
  - 1999: Stanton conjectured **monotonicity:** $c_{a+1}(n) \geq c_a(n)$
  - 2012: R. Nath & I conjectured **monotonicity:** $sc_{a+2}(n) \geq sc_a(n)$
- **Modular forms:** g.f. related to Dedekind's η-fcn, a m.f. of wt. 1/2.
- **Group Theory:** By Lascoux 2001, a-cores $\longleftrightarrow$ coset reps in $\widetilde{S}_a/S_a$
  Group actions on combinatorial objects!!!!

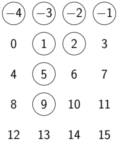


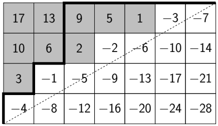
Anderson's bijection and the formula

Building on James's abacus diagrams, Anderson found a bijection:

$$\left\{ \begin{array}{c} \text{simultaneous} \\ (a,b)\text{-cores} \end{array} \right\} \overset{\textit{James}}{\longleftrightarrow} \left\{ \begin{array}{c} (a,b)\text{-flush} \\ \text{abaci} \end{array} \right\} \overset{\textit{And.}}{\longleftrightarrow} \left\{ \begin{array}{c} (a,b)\text{-Dyck paths} \\ (0,0) \to (b,a) \\ \text{above } y = \frac{a}{b}x \end{array} \right\}$$



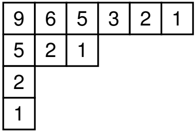


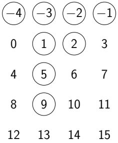


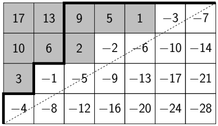
Anderson's bijection and the formula

Building on James's abacus diagrams, Anderson found a bijection:

$$\left\{ \begin{array}{c} \text{simultaneous} \\ (a,b)\text{-cores} \end{array} \right\} \overset{James}{\longleftrightarrow} \left\{ \begin{array}{c} (a,b)\text{-flush} \\ \text{abaci} \end{array} \right\} \overset{And.}{\longleftrightarrow} \left\{ \begin{array}{c} (a,b)\text{-Dyck paths} \\ (0,0) \to (b,a) \\ \text{above } y = \frac{a}{b}x \end{array} \right\}$$







Proof that the number of (a,b)-Dyck paths is $\frac{1}{a+b}\binom{a+b}{a}$:

- Path rotation gives an equivalence relation on the set of all lattice paths from $(0,0) \to (b,a)$.
- There are $\binom{a+b}{a}$ such paths and the equivalence classes have $a+b$ elements each.

Familiar numbers

a	1	2	3	4	5	6	
# of $(a, a+1)$-cores:							

Familiar numbers

a	1	2	3	4	5	6	n
# of $(a, a+1)$-cores:	1	2	5	14	42	132	

Familiar numbers

a	1	2	3	4	5	6	n
# of $(a, a+1)$-cores:	1	2	5	14	42	132	C_n

Familiar numbers

a	1	2	3	4	5	6	n
# of $(a, a+1)$-cores:	1	2	5	14	42	132	C_n

Specialize Anderson's result:

$(t, t+1)$-cores

$$\frac{1}{2t+1}\binom{2t+1}{t}$$

Familiar numbers

a	1	2	3	4	5	6	n
# of $(a, a+1)$-cores:	1	2	5	14	42	132	C_n

Specialize Anderson's result:

$\#\ (t, t+1)$-cores

$$\frac{1}{2t+1}\binom{2t+1}{t} = \frac{1}{t+1}\binom{2t}{t}$$

Familiar numbers

a	1	2	3	4	5	6	n
# of $(a, a+1)$-cores:	1	2	5	14	42	132	C_n

Specialize Anderson's result: # $(t, t+1)$-cores $\frac{1}{2t+1}\binom{2t+1}{t} = \frac{1}{t+1}\binom{2t}{t}$

Question: Is there a simple statistic on simultaneous core partitions that gives us a q-analog of the Catalan numbers?

$$\sum_{\substack{\lambda \text{ is a} \\ (t,t+1)\text{-core}}} q^{\text{stat}(\lambda)} = \frac{1}{[t+1]_q} \begin{bmatrix} 2t \\ t \end{bmatrix}_q$$

Familiar numbers

a	1	2	3	4	5	6	n
# of $(a, a+1)$-cores:	1	2	5	14	42	132	C_n

Specialize Anderson's result: # $(t, t+1)$-cores $\frac{1}{2t+1}\binom{2t+1}{t} = \frac{1}{t+1}\binom{2t}{t}$

Question: Is there a simple statistic on simultaneous core partitions that gives us a q-analog of the Catalan numbers?

$$\sum_{\substack{\lambda \text{ is a} \\ (t, t+1)\text{-core}}} q^{\mathrm{stat}(\lambda)} = \frac{1}{[t+1]_q} \begin{bmatrix} 2t \\ t \end{bmatrix}_q$$

Answer: Yes. We will create an analog of the **major statistic**.

The major statistic

For a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$, the **major statistic** $\mathrm{maj}(\pi)$ is the sum of the positions of the descents of π:

$$\mathrm{maj}(\pi) = \sum_{i : \pi_{i-1} > \pi_i} i.$$

The major statistic

For a permutation $\pi = \pi_1\pi_2\cdots\pi_n$, the **major statistic** $\mathrm{maj}(\pi)$ is the sum of the positions of the descents of π:

$$\mathrm{maj}(\pi) = \sum_{i:\pi_{i-1}>\pi_i} i.$$

For a $(t, t+1)$-core λ, create the sequence $b = (b_0, \ldots, b_{t-1})$, where $b_i = \#\ 1^{\text{st}}$ col. boxes with hook length $\equiv i \bmod t$.

Define $$\mathrm{maj}(\lambda) = \sum_{i\,:\,b_{i-1}\geq b_i} (2i - b_i).$$

See: maj defined as a sum over descents in a sequence.

The major statistic

For a permutation $\pi = \pi_1\pi_2\cdots\pi_n$, the **major statistic** $\mathrm{maj}(\pi)$ is the sum of the positions of the descents of π:

$$\mathrm{maj}(\pi) = \sum_{i:\pi_{i-1}>\pi_i} i.$$

For a $(t, t+1)$-core λ, create the sequence $b = (b_0, \ldots, b_{t-1})$, where $b_i = \#\ 1^{\text{st}}$ col. boxes with hook length $\equiv i \bmod t$.

Define $$\mathrm{maj}(\lambda) = \sum_{i\,:\,b_{i-1}\geq b_i} (2i - b_i).$$

Theorem. (AHJ '13)

$$\sum_{\substack{\lambda \text{ is a} \\ (t,\,t+1)\text{-core}}} q^{\mathrm{maj}(\lambda)} = \frac{1}{[t+1]_q}\begin{bmatrix}2t\\ t\end{bmatrix}_q$$

See: maj defined as a sum over descents in a sequence.

The major statistic

For a permutation $\pi = \pi_1\pi_2\cdots\pi_n$, the **major statistic** $\mathrm{maj}(\pi)$ is the sum of the positions of the descents of π:

$$\mathrm{maj}(\pi) = \sum_{i:\pi_{i-1}>\pi_i} i.$$

For a $(t, t+1)$-core λ, create the sequence $b = (b_0, \ldots, b_{t-1})$, where $b_i = \#\ 1^{\text{st}}$ col. boxes with hook length $\equiv i \bmod t$.

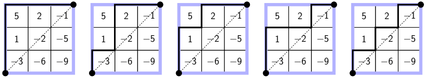
Define $$\mathrm{maj}(\lambda) = \sum_{i\,:\,b_{i-1}\geq b_i} (2i - b_i).$$

Theorem. (AHJ '13)

$$\sum_{\substack{\lambda \text{ is a} \\ (t,\,t+1)\text{-core}}} q^{\mathrm{maj}(\lambda)} = \frac{1}{[t+1]_q} \begin{bmatrix} 2t \\ t \end{bmatrix}_q$$

See: maj defined as a sum over descents in a sequence.

Why? Major index on Dyck paths!



Add positions of valleys: $\frac{1}{[4]_q}\begin{bmatrix} 6 \\ 3 \end{bmatrix}_q = q^0 + q^2 + q^3 + q^4 + q^{2+4}$

Reflection Groups

The combinatorics of **groups**:

- Made up of a set of elements $W = \{w_1, w_2, \ldots\}$.
- Multiplication of two elements $w_1 w_2$ stays in the group.

Reflection Groups

The combinatorics of **groups**:

- Made up of a set of elements $W = \{w_1, w_2, \dots\}$.
- Multiplication of two elements $w_1 w_2$ stays in the group.
  - ALTHOUGH, it is **not** the case that $w_1 w_2 = w_2 w_1$.

Reflection Groups

The combinatorics of **groups**:

- Made up of a set of elements $W = \{w_1, w_2, \ldots\}$.
- Multiplication of two elements $w_1 w_2$ stays in the group.
  - ALTHOUGH, it is **not** the case that $w_1 w_2 = w_2 w_1$.
- There is an identity element (id) & Every element has an inverse.
- Think: (Non-zero real numbers) or (invertible $n \times n$ matrices.)

Reflection Groups

The combinatorics of **groups**:

- Made up of a set of elements $W = \{w_1, w_2, \ldots\}$.
- Multiplication of two elements $w_1 w_2$ stays in the group.
  - ALTHOUGH, it is **not** the case that $w_1 w_2 = w_2 w_1$.
- There is an identity element (id) & Every element has an inverse.
- Think: (Non-zero real numbers) or (invertible $n \times n$ matrices.)

We will talk about **reflection groups**. (With nice pictures)

- W is **generated** by a set of generators $S = \{s_1, s_2, \ldots, s_k\}$.
- Along with a set of **relations**.

Reflection Groups

The combinatorics of **groups**:

- Made up of a set of elements $W = \{w_1, w_2, \ldots\}$.
- Multiplication of two elements $w_1 w_2$ stays in the group.
  - ALTHOUGH, it is **not** the case that $w_1 w_2 = w_2 w_1$.
- There is an identity element (id) & Every element has an inverse.
- Think: (Non-zero real numbers) or (invertible $n \times n$ matrices.)

We will talk about **reflection groups**. (With nice pictures)

- W is **generated** by a set of generators $S = \{s_1, s_2, \ldots, s_k\}$.
  - Every $w \in W$ can be written as a product of generators.
- Along with a set of **relations**.

Reflection Groups

The combinatorics of **groups**:

- Made up of a set of elements $W = \{w_1, w_2, \ldots\}$.
- Multiplication of two elements $w_1 w_2$ stays in the group.
  - ALTHOUGH, it is **not** the case that $w_1 w_2 = w_2 w_1$.
- There is an identity element (id) & Every element has an inverse.
- Think: (Non-zero real numbers) or (invertible $n \times n$ matrices.)

We will talk about **reflection groups**. (With nice pictures)

- W is **generated** by a set of generators $S = \{s_1, s_2, \ldots, s_k\}$.
  - Every $w \in W$ can be written as a product of generators.
- Along with a set of **relations**.
  - These are rules to convert between expressions.
  - $s_i^2 = \text{id}$. —and— $(s_i s_j)^{\text{power}} = \text{id}$.

Reflection Groups

The combinatorics of **groups**:

- Made up of a set of elements $W = \{w_1, w_2, \ldots\}$.
- Multiplication of two elements $w_1 w_2$ stays in the group.
  - ALTHOUGH, it is **not** the case that $w_1 w_2 = w_2 w_1$.
- There is an identity element (id) & Every element has an inverse.
- Think: (Non-zero real numbers) or (invertible $n \times n$ matrices.)

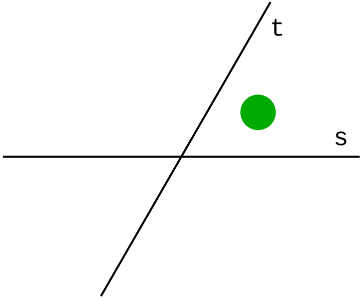
We will talk about **reflection groups**. (With nice pictures)

- W is **generated** by a set of generators $S = \{s_1, s_2, \ldots, s_k\}$.
  - Every $w \in W$ can be written as a product of generators.
- Along with a set of **relations**.
  - These are rules to convert between expressions.
  - $s_i^2 = \text{id}$. —and— $(s_i s_j)^{\text{power}} = \text{id}$.

For example, $w = s_3 s_2 s_1 s_1 s_2 s_4 = s_3 s_2 \text{id} s_2 s_4 = s_3 \text{id} s_4 = s_3 s_4$

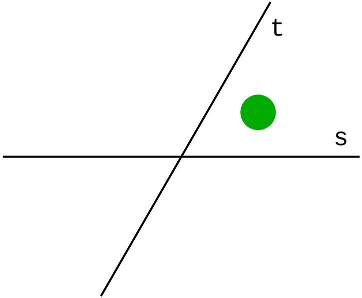
Reflection Groups

- The action of multiplying (on the left) by a generator s corresponds to a reflection across a hyperplane H_s. $(s_i^2 = \mathrm{id})$



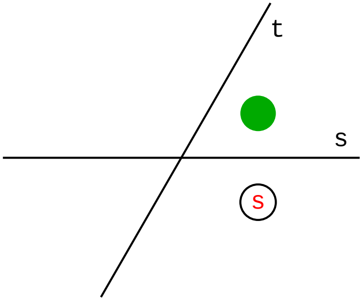
Reflection Groups

- The action of multiplying (on the left) by a generator s corresponds to a reflection across a hyperplane H_s. $(s_i^2 = \text{id})$



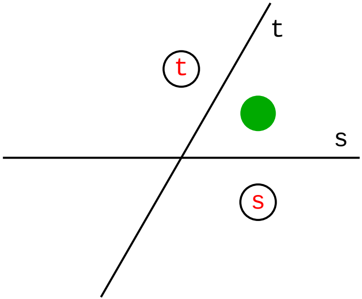
Reflection Groups

- The action of multiplying (on the left) by a generator s corresponds to a reflection across a hyperplane H_s. $(s_i^2 = \mathrm{id})$



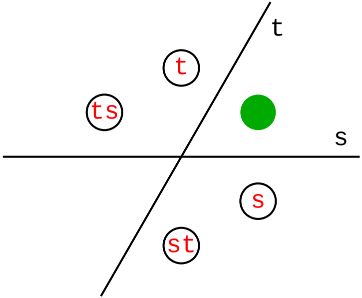
Reflection Groups

- The action of multiplying (on the left) by a generator s corresponds to a reflection across a hyperplane H_s. $(s_i^2 = \text{id})$



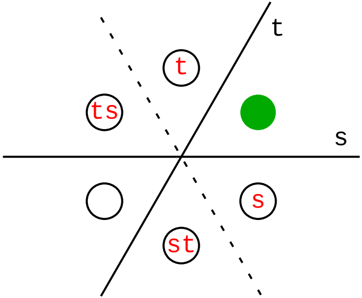
Reflection Groups

- The action of multiplying (on the left) by a generator s corresponds to a reflection across a hyperplane H_s. ($s_i^2 = \mathrm{id}$)



Reflection Groups

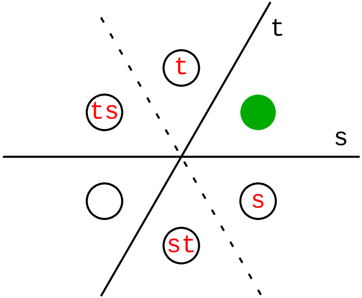
- The action of multiplying (on the left) by a generator s corresponds to a reflection across a hyperplane H_s. $(s_i^2 = \mathrm{id})$



- When the angle between H_s and H_t is $\frac{\pi}{3}$, relation is $(st)^3 = \mathrm{id}$.

Reflection Groups

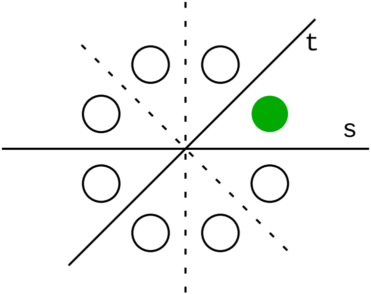
- The action of multiplying (on the left) by a generator s corresponds to a reflection across a hyperplane H_s. ($s_i^2 = \mathrm{id}$)



- When the angle between H_s and H_t is $\frac{\pi}{3}$, relation is $(st)^3 = \mathrm{id}$.
- The group depends on the placement of the hyperplanes. $|S| = 6$.

Reflection Groups

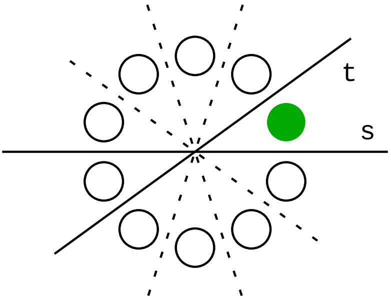
- The action of multiplying (on the left) by a generator s corresponds to a reflection across a hyperplane H_s. $(s_i^2 = \mathrm{id})$



- When the angle between H_s and H_t is $\frac{\pi}{4}$, relation is $(st)^4 = \mathrm{id}$.
- The group depends on the placement of the hyperplanes. $|S| = 8$.

Reflection Groups

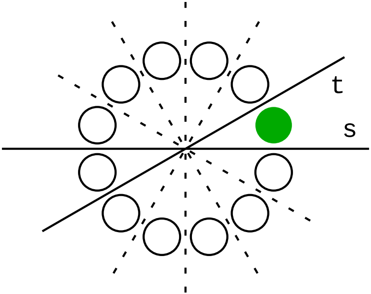
- The action of multiplying (on the left) by a generator s corresponds to a reflection across a hyperplane H_s. $(s_i^2 = \text{id})$



- When the angle between H_s and H_t is $\frac{\pi}{5}$, relation is $(st)^5 = \text{id}$.
- The group depends on the placement of the hyperplanes. $|S| = 10$.

Reflection Groups

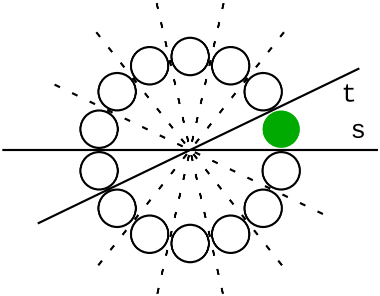
- The action of multiplying (on the left) by a generator s corresponds to a reflection across a hyperplane H_s. ($s_i^2 = \mathrm{id}$)



- When the angle between H_s and H_t is $\frac{\pi}{6}$, relation is $(st)^6 = \mathrm{id}$.
- The group depends on the placement of the hyperplanes. $|S| = 12$.

Reflection Groups

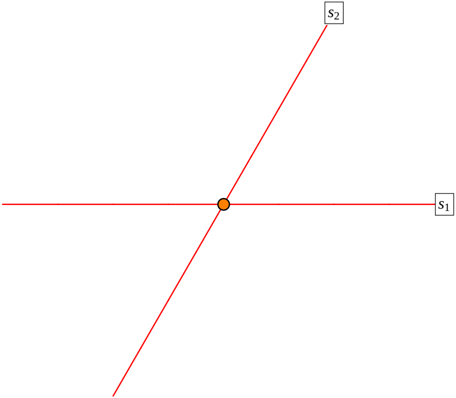
- The action of multiplying (on the left) by a generator s corresponds to a reflection across a hyperplane H_s. ($s_i^2 = \text{id}$)



- When the angle between H_s and H_t is $\frac{\pi}{n}$, relation is $(st)^n = \text{id}$.
- The group depends on the placement of the hyperplanes. $|S| = 2n$.

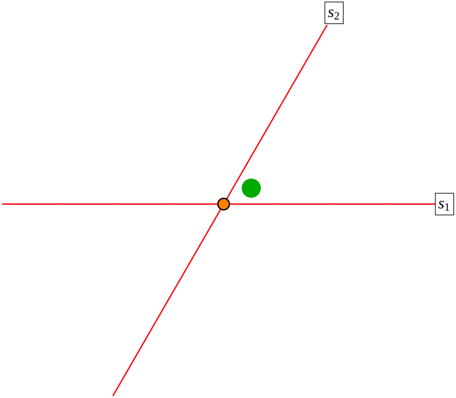
Infinite Reflection Groups

An infinite reflection group: the **affine permutations** $\widetilde{S}_n$.



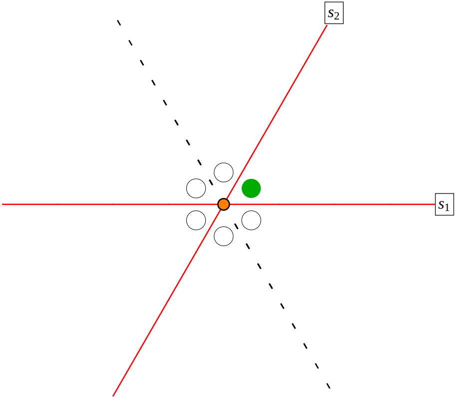
Infinite Reflection Groups

An infinite reflection group: the **affine permutations** $\widetilde{S}_n$.



Infinite Reflection Groups

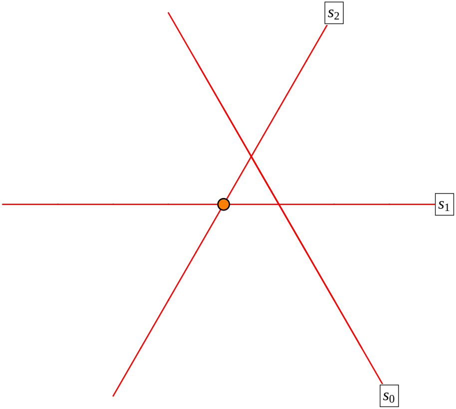
An infinite reflection group: the **affine permutations** $\widetilde{S}_n$.



Infinite Reflection Groups

An infinite reflection group: the **affine permutations** $\widetilde{S}_n$.

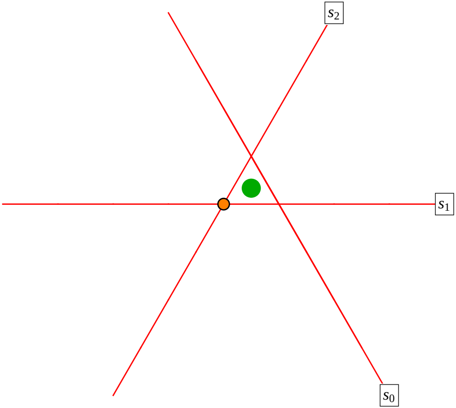
- Add a new generator s_0 and a new affine hyperplane H_0.



Infinite Reflection Groups

An infinite reflection group: the **affine permutations** $\widetilde{S}_n$.

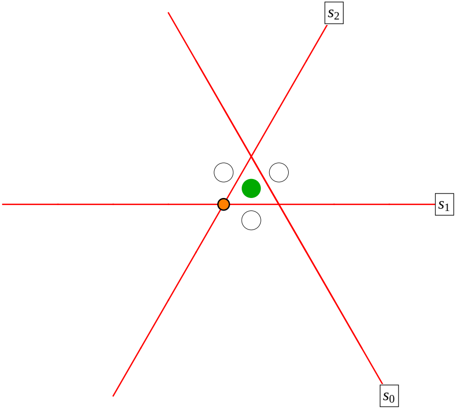
- Add a new generator s_0 and a new affine hyperplane H_0.



Infinite Reflection Groups

An infinite reflection group: the **affine permutations** $\widetilde{S}_n$.

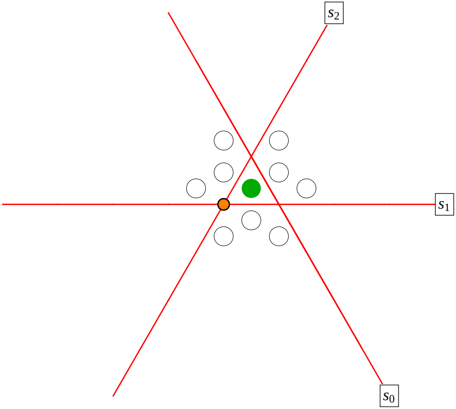
- Add a new generator s_0 and a new affine hyperplane H_0.



Infinite Reflection Groups

An infinite reflection group: the **affine permutations** $\widetilde{S}_n$.

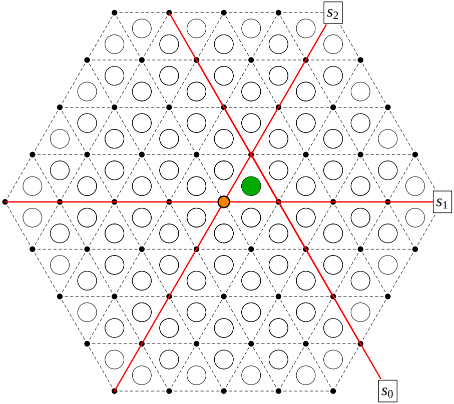
- Add a new generator s_0 and a new affine hyperplane H_0.



Infinite Reflection Groups

An infinite reflection group: the **affine permutations** $\widetilde{S}_n$.

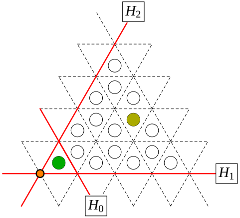
- Add a new generator s_0 and a new affine hyperplane H_0.



Elements generated by $\{s_0, s_1, s_2\}$ correspond to **alcoves** here.

Combinatorics of affine permutations

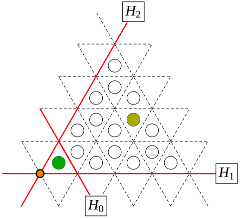
Many ways to reference elements in $\widetilde{S}_n$.



Combinatorics of affine permutations

Many ways to reference elements in $\widetilde{S}_n$.

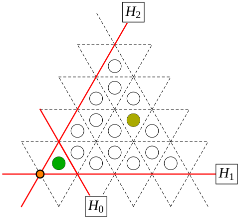
- **Geometry.** Point to the alcove.



Combinatorics of affine permutations

Many ways to reference elements in $\widetilde{S}_n$.

- **Geometry.** Point to the alcove.
- **Alcove coordinates.** Keep track of how many hyperplanes of each type you have crossed to get to your alcove.



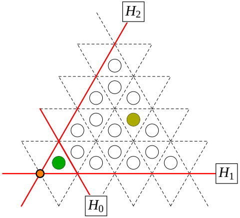
Coordinates:

3	1
1	

Combinatorics of affine permutations

Many ways to reference elements in $\widetilde{S}_n$.

- **Geometry.** Point to the alcove.
- **Alcove coordinates.** Keep track of how many hyperplanes of each type you have crossed to get to your alcove.
- **Word.** Write the element as a (short) product of generators.



Coordinates:

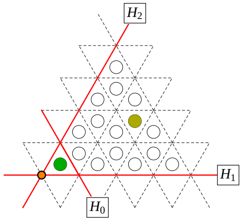
3	1
1	

Word: $s_0 s_1 s_2 s_1 s_0$

Combinatorics of affine permutations

Many ways to reference elements in $\widetilde{S}_n$.

- **Geometry.** Point to the alcove.
- **Alcove coordinates.** Keep track of how many hyperplanes of each type you have crossed to get to your alcove.
- **Word.** Write the element as a (short) product of generators.
- **Permutation.** Similar to writing finite permutations as 312.



Coordinates:

3	1
1	

Word: $s_0 s_1 s_2 s_1 s_0$

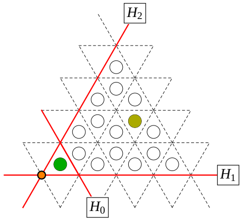
Permutation:

$$(-3, 2, 7)$$

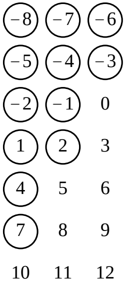
Combinatorics of affine permutations

Many ways to reference elements in $\widetilde{S}_n$.

- **Geometry.** Point to the alcove.
- **Alcove coordinates.** Keep track of how many hyperplanes of each type you have crossed to get to your alcove.
- **Word.** Write the element as a (short) product of generators.
- **Permutation.** Similar to writing finite permutations as 312.
- **Abacus diagram.** Columns of numbers.



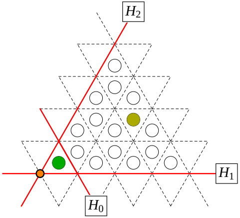
Abacus diagram:



Combinatorics of affine permutations

Many ways to reference elements in $\widetilde{S}_n$.

- **Geometry.** Point to the alcove.
- **Alcove coordinates.** Keep track of how many hyperplanes of each type you have crossed to get to your alcove.
- **Word.** Write the element as a (short) product of generators.
- **Permutation.** Similar to writing finite permutations as 312.
- **Abacus diagram.** Columns of numbers.
- **Core partition.** Hook length condition.



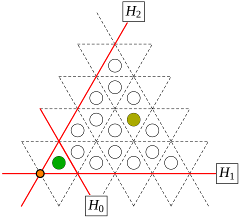
Core partition:

0	*1*	*2*	*0*
2	*0*		
1			
0			

Combinatorics of affine permutations

Many ways to reference elements in $\widetilde{S}_n$.

- **Geometry.** Point to the alcove.
- **Alcove coordinates.** Keep track of how many hyperplanes of each type you have crossed to get to your alcove.
- **Word.** Write the element as a (short) product of generators.
- **Permutation.** Similar to writing finite permutations as 312.
- **Abacus diagram.** Columns of numbers.
- **Core partition.** Hook length condition.
- **Bounded partition.** Part size bounded.



Core partition:

0	1	2	0
2	0		
1			
0			

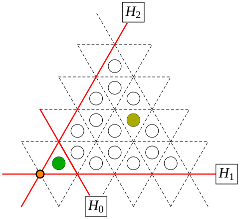
Bounded partition:

0	1
2	
1	
0	

Combinatorics of affine permutations

Many ways to reference elements in $\widetilde{S}_n$.

- **Geometry.** Point to the alcove.
- **Alcove coordinates.** Keep track of how many hyperplanes of each type you have crossed to get to your alcove.
- **Word.** Write the element as a (short) product of generators.
- **Permutation.** Similar to writing finite permutations as 312.
- **Abacus diagram.** Columns of numbers.
- **Core partition.** Hook length condition.
- **Bounded partition.** Part size bounded.
- **Others!** Lattice path, order ideal, etc.



Core partition:

0	1	2	0
2	0		
1			
0			

Bounded partition:

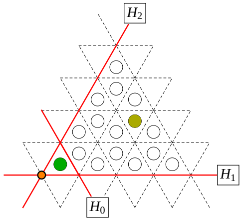
0	1
2	
1	
0	

Combinatorics of affine permutations

Many ways to reference elements in $\widetilde{S}_n$.

- **Geometry.** Point to the alcove.
- **Alcove coordinates.** Keep track of how many hyperplanes of each type you have crossed to get to your alcove.
- **Word.** Write the element as a (short) product of generators.
- **Permutation.** Similar to writing finite permutations as 312.
- **Abacus diagram.** Columns of numbers.
- **Core partition.** Hook length condition.
- **Bounded partition.** Part size bounded.
- **Others!** Lattice path, order ideal, etc.

They all play nicely with each other.



Core partition:

0	1	2	0
2	0		
1			
0			

Bounded partition:

0	1
2	
1	
0	

Action of generators on the core partition

- Label the boxes of λ with residues.
- s_i acts by adding or removing boxes with residue i.

0	1	2	3	0	1
3	0	1	2	3	0
2	3	0	1	2	3
1	2	3	0	1	2
0	1	2	3	0	1
3	0	1	2	3	0

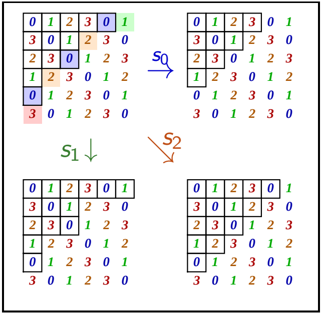
Action of generators on the core partition

- Label the boxes of λ with residues.
- s_i acts by adding or removing boxes with residue i.

0	1	2	3	0	1
3	0	1	2	3	0
2	3	0	1	2	3
1	2	3	0	1	2
0	1	2	3	0	1
3	0	1	2	3	0

Example. $\lambda = (5, 3, 3, 1, 1)$ is a 4-core.

- has removable 0 boxes
- has addable 1, 2, 3 boxes.



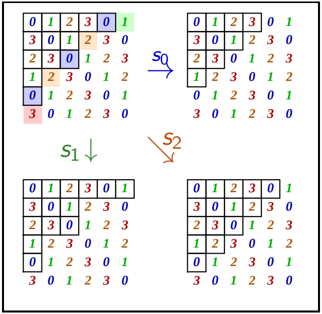
Action of generators on the core partition

- Label the boxes of λ with residues.
- s_i acts by adding or removing boxes with residue i.

0	1	2	3	0	1
3	0	1	2	3	0
2	3	0	1	2	3
1	2	3	0	1	2
0	1	2	3	0	1
3	0	1	2	3	0

Example. $\lambda = (5,3,3,1,1)$ is a 4-core.

- has removable 0 boxes
- has addable 1, 2, 3 boxes.

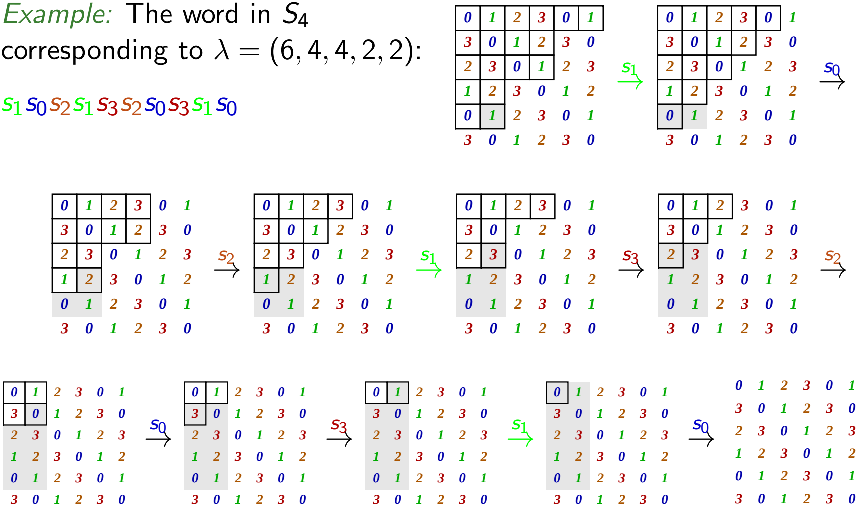


Idea: We can use this to figure out a *word* for w.

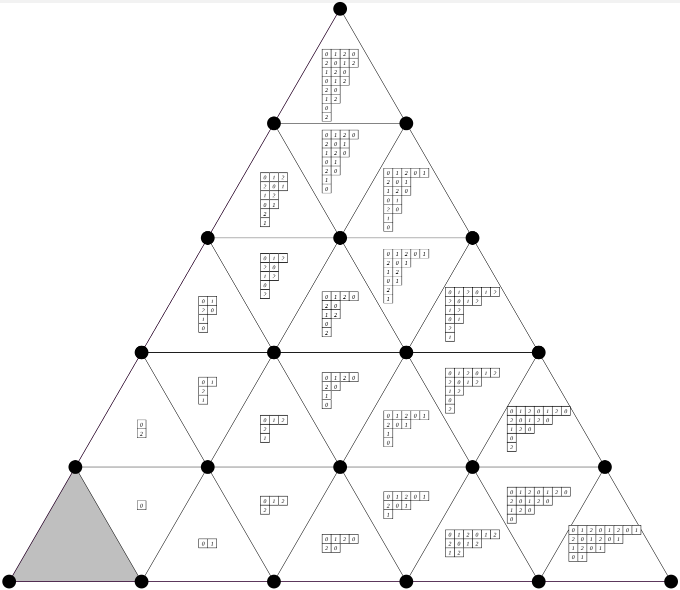
Finding a word for an affine permutation.

Example: The word in S_4 corresponding to $\lambda = (6,4,4,2,2)$:

$s_1 s_0 s_2 s_1 s_3 s_2 s_0 s_3 s_1 s_0$

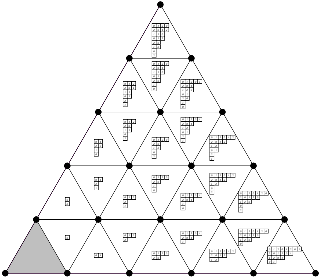


The bijection between cores and alcoves



Simultaneous core partitions

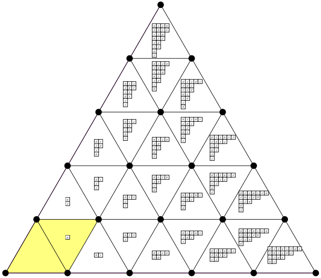
Fishel–Vazirani proved an alcove interpretation of $n/(mn+1)$-cores.



Simultaneous core partitions

Fishel–Vazirani proved an alcove interpretation of $n/(mn+1)$-cores.

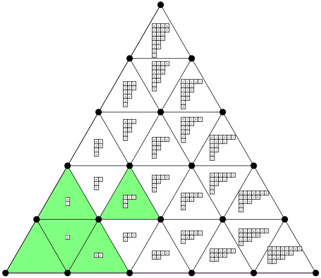
How many partitions are both 3-cores and 2-cores? $C_2 = 2$.



Simultaneous core partitions

Fishel–Vazirani proved an alcove interpretation of $n/(mn+1)$-cores.

How many partitions are both 3-cores and 2-cores? $C_2 = 2$.

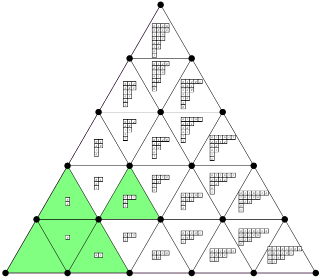


How many partitions are both 3-cores and 4-cores? $C_3 = 5$.

Simultaneous core partitions

Fishel–Vazirani proved an alcove interpretation of $n/(mn+1)$-cores.

How many partitions are both 3-cores and 2-cores? $C_2 = 2$.



How many partitions are both 3-cores and 4-cores? $C_3 = 5$.

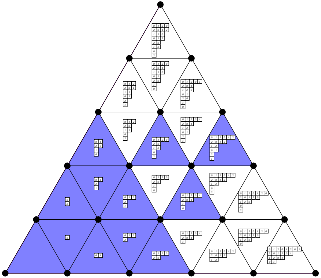
From Anderson's formula $\frac{1}{a+b}\binom{a+b}{a}$,

The number of $(3,7)$-cores is $\frac{1}{10}\binom{10}{3} = \frac{1}{10}\frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1} = 12$.

Simultaneous core partitions

Fishel–Vazirani proved an alcove interpretation of $n/(mn+1)$-cores.

How many partitions are both 3-cores and 2-cores? $C_2 = 2$.



How many partitions are both 3-cores and 4-cores? $C_3 = 5$.

From Anderson's formula $\frac{1}{a+b}\binom{a+b}{a}$,

The number of $(3,7)$-cores is $\frac{1}{10}\binom{10}{3} = \frac{1}{10}\frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1} = 12$.

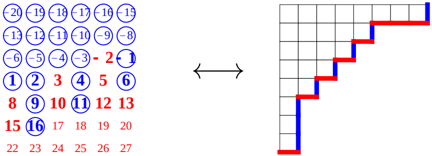
Research Questions

★ Can we extend combinatorial interps to other reflection groups?

Research Questions

★ Can we extend combinatorial interps to other reflection groups?

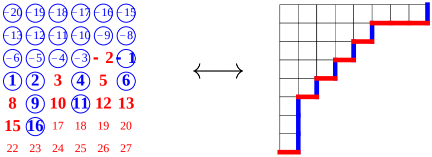
- Yes! Involves self-conjugate partitions.



Research Questions

★ Can we extend combinatorial interps to other reflection groups?

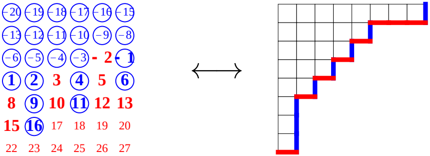
- Yes! Involves self-conjugate partitions.
- Article (28 pp) published in *Journal of Algebra.* (2012)
  Sets up the theory.
- Joint with Brant Jones, JMU,



Research Questions

★ Can we extend combinatorial interps to other reflection groups?

- Yes! Involves self-conjugate partitions.
- Article (28 pp) published in *Journal of Algebra.* (2012)
  Sets up the theory.
- Article (16 pp) published in *European Journal of Comb.*
  Applies the theory.
- Joint with Brant Jones, JMU, Drew Armstrong, Miami.



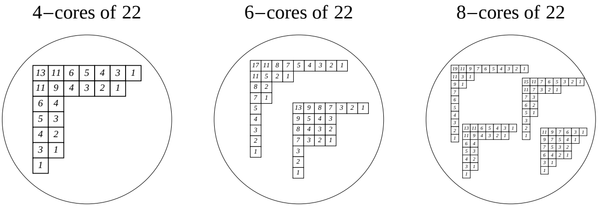
Research Questions

★ What numerical properties do self-conjugate core partitions have?

Research Questions

★ What numerical properties do self-conjugate core partitions have?

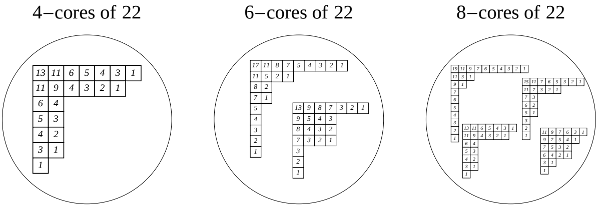
- There are more (s.c. $t+2$-cores of n) than (s.c. t-cores of n).



Research Questions

★ What numerical properties do self-conjugate core partitions have?

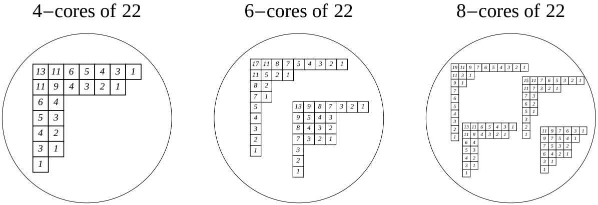
- There are more (s.c. $t+2$-cores of n) than (s.c. t-cores of n).
- Article (17 pp) published in *Journal of Number Theory.* (2013)



Research Questions

★ What numerical properties do self-conjugate core partitions have?

- There are more (s.c. $t+2$-cores of n) than (s.c. t-cores of n).
- Article (17 pp) published in *Journal of Number Theory.* (2013)
- Joint with Rishi Nath, York College, CUNY.



Research Questions

★ What numerical properties do self-conjugate core partitions have?

- There are more (s.c. $t+2$-cores of n) than (s.c. t-cores of n).
- Article (17 pp) published in *Journal of Number Theory.* (2013)
- Joint with Rishi Nath, York College, CUNY.

★ Properties of simultaneous core partitions. $\left(\text{Formula: } \frac{1}{s+t}\binom{s+t}{s}\right)$

Research Questions

- ★ What numerical properties do self-conjugate core partitions have?
  - There are more (s.c. $t+2$-cores of n) than (s.c. t-cores of n).
  - Article (17 pp) published in *Journal of Number Theory.* (2013)
  - Joint with Rishi Nath, York College, CUNY.
- ★ Properties of simultaneous core partitions. $\left(\text{Formula: } \frac{1}{s+t}\binom{s+t}{s}\right)$
  - **Question.** What is the average size of an (s, t)-core partition?

Research Questions

★ What numerical properties do self-conjugate core partitions have?

- There are more (s.c. $t+2$-cores of n) than (s.c. t-cores of n).
- Article (17 pp) published in *Journal of Number Theory.* (2013)
- Joint with Rishi Nath, York College, CUNY.

★ Properties of simultaneous core partitions. (Formula: $\frac{1}{s+t}\binom{s+t}{s}$)

- **Question.** What is the average size of an (s,t)-core partition?
- *Progress:* Answer: $(s+t+1)(s-1)(t-1)/24$. Proof?

Research Questions

★ What numerical properties do self-conjugate core partitions have?

- There are more (s.c. $t+2$-cores of n) than (s.c. t-cores of n).
- Article (17 pp) published in *Journal of Number Theory.* (2013)
- Joint with Rishi Nath, York College, CUNY.

★ Properties of simultaneous core partitions. $\left(\text{Formula: } \frac{1}{s+t}\binom{s+t}{s}\right)$

- **Question.** What is the average size of an (s,t)-core partition?
- *Progress:* Answer: $(s+t+1)(s-1)(t-1)/24$. Proof?
- **Question:** Is there a core statistic for a q-analog of $\frac{1}{s+t}\binom{s+t}{s}$?

Research Questions

★ What numerical properties do self-conjugate core partitions have?

- There are more (s.c. $t+2$-cores of n) than (s.c. t-cores of n).
- Article (17 pp) published in *Journal of Number Theory.* (2013)
- Joint with Rishi Nath, York College, CUNY.

★ Properties of simultaneous core partitions. $\left(\text{Formula: } \frac{1}{s+t}\binom{s+t}{s}\right)$

- **Question.** What is the average size of an (s,t)-core partition?
- *Progress:* Answer: $(s+t+1)(s-1)(t-1)/24$. Proof?
- **Question:** Is there a core statistic for a q-analog of $\frac{1}{s+t}\binom{s+t}{s}$?
- *Progress:* m-Catalan number C_3 through $(3,3m+1)$-cores.

Research Questions

★ What numerical properties do self-conjugate core partitions have?

- There are more (s.c. $t+2$-cores of n) than (s.c. t-cores of n).
- Article (17 pp) published in *Journal of Number Theory.* (2013)
- Joint with Rishi Nath, York College, CUNY.

★ Properties of simultaneous core partitions. $\left(\text{Formula: } \frac{1}{s+t}\binom{s+t}{s}\right)$

- **Question.** What is the average size of an (s,t)-core partition?
- *Progress:* Answer: $(s+t+1)(s-1)(t-1)/24$. Proof?
- **Question:** Is there a core statistic for a q-analog of $\frac{1}{s+t}\binom{s+t}{s}$?
- *Progress:* m-Catalan number C_3 through $(3,3m+1)$-cores.
- **Question:** Why is the zeta map a bijection?

- Joint with Tom Denton and Cesar Ceballos.

Research Questions

★ What numerical properties do self-conjugate core partitions have?

- There are more (s.c. $t+2$-cores of n) than (s.c. t-cores of n).
- Article (17 pp) published in *Journal of Number Theory.* (2013)
- Joint with Rishi Nath, York College, CUNY.

★ Properties of simultaneous core partitions. (Formula: $\frac{1}{s+t}\binom{s+t}{s}$)

- **Question.** What is the average size of an (s,t)-core partition?
- *Progress:* Answer: $(s+t+1)(s-1)(t-1)/24$. Proof?
- **Question:** Is there a core statistic for a q-analog of $\frac{1}{s+t}\binom{s+t}{s}$?
- *Progress:* m-Catalan number C_3 through $(3,3m+1)$-cores.
- **Question:** Why is the zeta map a bijection?
- *Progress:* An inverse in certain cases.
- Joint with Tom Denton and Cesar Ceballos.

Research Questions

★ What numerical properties do self-conjugate core partitions have?

- There are more (s.c. $t+2$-cores of n) than (s.c. t-cores of n).
- Article (17 pp) published in *Journal of Number Theory.* (2013)
- Joint with Rishi Nath, York College, CUNY.

★ Properties of simultaneous core partitions. (Formula: $\frac{1}{s+t}\binom{s+t}{s}$)

- **Question.** What is the average size of an (s,t)-core partition?
- *Progress:* Answer: $(s+t+1)(s-1)(t-1)/24$. Proof?
- **Question:** Is there a core statistic for a q-analog of $\frac{1}{s+t}\binom{s+t}{s}$?
- *Progress:* m-Catalan number C_3 through $(3, 3m+1)$-cores.
- **Question:** Why is the zeta map a bijection?
- *Progress:* An inverse in certain cases.
- Joint with Tom Denton and Cesar Ceballos.

★ Happy to have students who would like to do research!

Course Evaluation

Please comment on:

- Prof. Chris's effectiveness as a teacher.
- Prof. Chris's contribution to your learning.
- The course material: What you enjoyed and/or found challenging.
- Is there anything you would change about the course?
- How did the reality of the course compare to your expectations?
- Is there anything else Prof. Chris should know?

Place completed evaluations in the provided folder.