Combinatorics of Core Partitions

Christopher R. H. Hanusa
Queens College, CUNY

Joint work with Brant Jones, James Madison University
Drew Armstrong, University of Miami
Rishi Nath, York College, CUNY
Tom Denton, Google
Cesar Ceballos, York University, Toronto

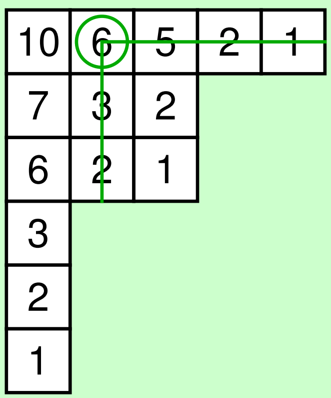
qc.edu/~chanusa/research

Partitions

The **Young diagram** of $\lambda = (\lambda_1, \dots, \lambda_k)$ has λ_i boxes in row i.

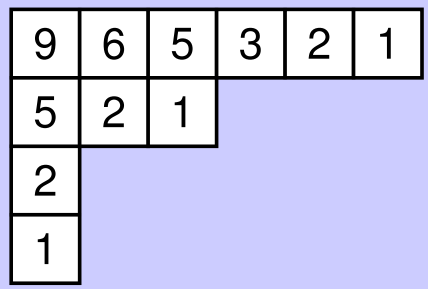
The **hook length** of a box = # boxes below + # boxes to right + box

λ is an ***a*-core** if no boxes have hook length a.



4-Core Partition

$\lambda = (5, 3, 3, 1, 1, 1)$



Simultaneous

$(4, 7)$-core partition

- There are infinitely many a-core partitions. $(a \geq 2)$

Of interest: Partitions that are **both** a-core **and** b-core. $(a, b) = 1$

- (Anderson, 2002): # (a, b)-core partitions equals $\frac{1}{a+b}\binom{a+b}{a}$.

Partitions and Abacus Diagrams

An **abacus diagram** is a function $\mathcal{A} : \mathbb{Z} \to \{\bullet, \text{␣}\}$.

(−5) (−4) (−3) (−2) (−1) 0 (1) (2) (3) 4 5 (6) (7) 8 9 (10) 11 12 13

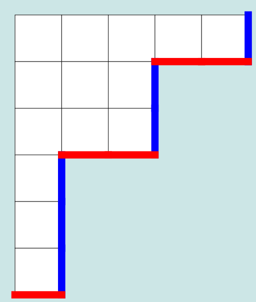
(Equivalence class...)

Bijection!

a-core partitions	$\longleftrightarrow$	*a*-flush abacus diagrams

Rule: Read the abacus from the boundary of λ.

- vertical step $\leftrightarrow$ bead
- horizontal step $\leftrightarrow$ gap



(−5) (−4) (−3) (−2) (−1) 0 (1) (2) (3) 4 5 (6) (7) 8 9 (10) 11 12 13

Normalized

(−8)	(−7)	(−6)	(−5)
(−4)	(−3)	(−2)	(−1)
0	(1)	(2)	(3)
4	5	(6)	(7)
8	9	(10)	11

Balanced

(−7)	(−6)	(**-5**)	(−4)
(−3)	(−2)	−1	(**0**)
(1)	(2)	3	4
(5)	(**6**)	7	8
(**9**)	10	11	12

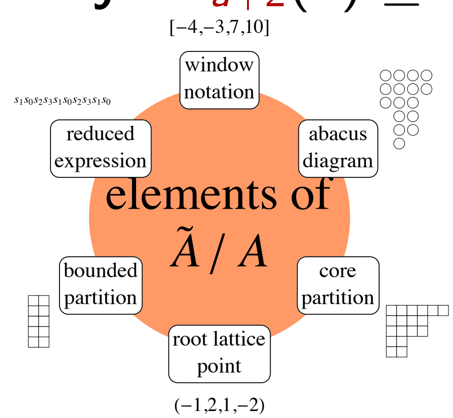
Core partitions in the literature

- **Representation Theory:** (origin)
  - Nakayama conjecture, proved by Brauer & Robinson 1947 says a-cores label a-blocks of irreducible modular representations for S_n.
- **Number Theory:**
  - Let $c_a(n) = \#$ of a-core partitions of n.
  - In 1976, Olsson proved $\displaystyle\sum_{n\geq 0} c_a(n)x^n = \prod_{n\geq 1} \frac{(1-x^{na})^a}{1-x^n}$

  Numerical properties of $c_a(n)$?

  - 1996: Granville & Ono proved **positivity:** $c_a(n) > 0$ $(a \geq 4)$.
  - 1999: Stanton conjectured **monotonicity:** $c_{a+1}(n) \geq c_a(n)$
  - 2012: R. Nath & I conjectured **monotonicity:** $sc_{a+2}(n) \geq sc_a(n)$
- **Modular forms:** g.f. related to Dedekind's η-fcn, a m.f. of wt. $1/2$.
- **Group Theory:** By Lascoux 2001, a-cores $\longleftrightarrow$ coset reps in $\widetilde{S}_a/S_a$

  Group actions on combinatorial objects!!!!



Affine permutations

(Finite) n-Permutations $\pi \in S_n$

- Write π in one-line notation. (e.g. 2 1 4 5 3 6)
- Write π as a product of *adjacent transpositions* $\{s_1, s_2, \ldots, s_{n-1}\}$
  - $s_i : (i) \leftrightarrow (i+1)$. (e.g. $s_4 = 1\,2\,3\,5\,4\,6$)
  - The word for 2 1 4 5 3 6 is $s_1 s_3 s_4$.

123	123
213	132
231	312
321	321

These **generators** interact:

- Consecutive generators don't commute: $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$
- Non-consecutive generators do commute: $s_i s_j = s_j s_i$.

Affine n-Permutations $\pi \in \widetilde{S}_n$

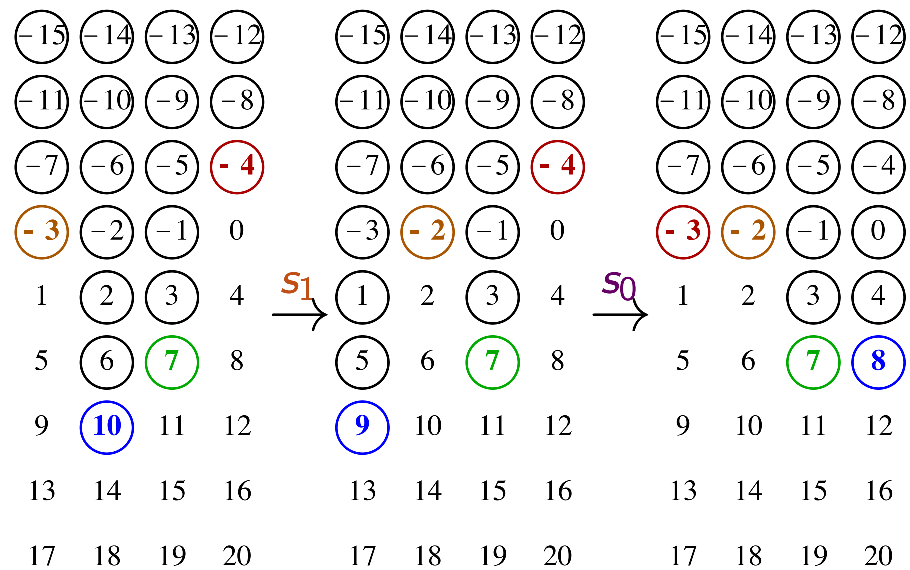
- Generators: $\{s_0, s_1, \ldots, s_{n-1}\}$
- Can think of as permutations of $\mathbb{Z}$.
- Window notation: $[-4, -3, 7, 10]$

Action of generators on abacus diagrams

(James and Kerber, 1981) Given an affine permutation $[w_1, \ldots, w_n]$,

- Create a balanced abacus on n runners where each runner has a lowest bead at w_i.

Example: $[-4, -3, 7, 10]$



- **Generators act nicely.**
- s_i interchanges runners $i \leftrightarrow i+1$. $(s_1 : 1 \leftrightarrow 2)$
- s_0 interchanges runners 1 and n (with shifts) $(s_0 : 1 \overset{\text{shift}}{\leftrightarrow} 4)$

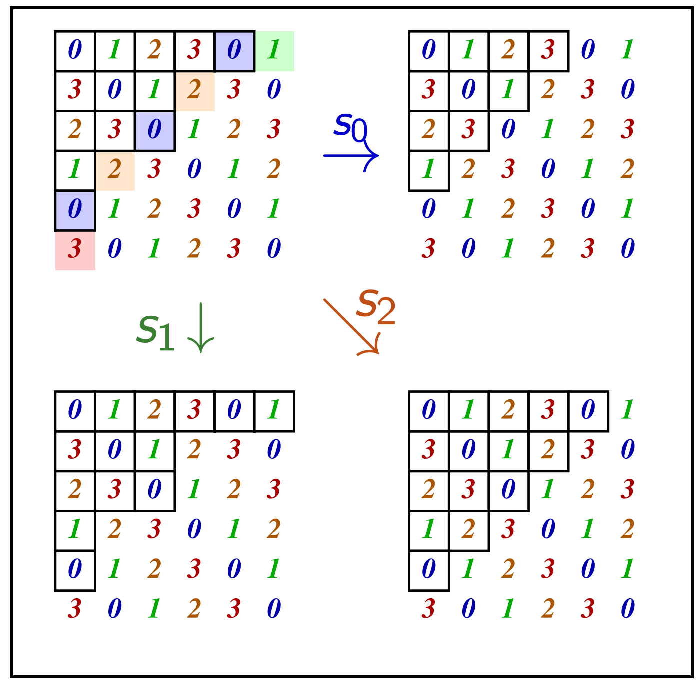
Action of generators on core partition

- Label the boxes of λ with residues.
- s_i acts by adding or removing boxes with residue i.

0	1	2	3	0	1
3	0	1	2	3	0
2	3	0	1	2	3
1	2	3	0	1	2
0	1	2	3	0	1
3	0	1	2	3	0

Example. $\lambda = (5, 3, 3, 1, 1)$ is a 4-core.

- has removable 0 boxes
- has addable 1, 2, 3 boxes.

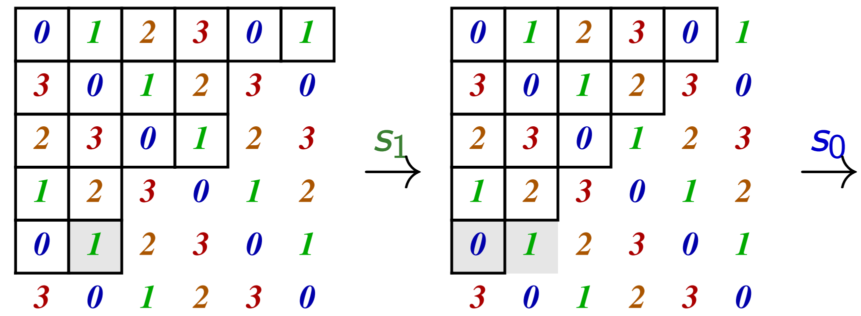


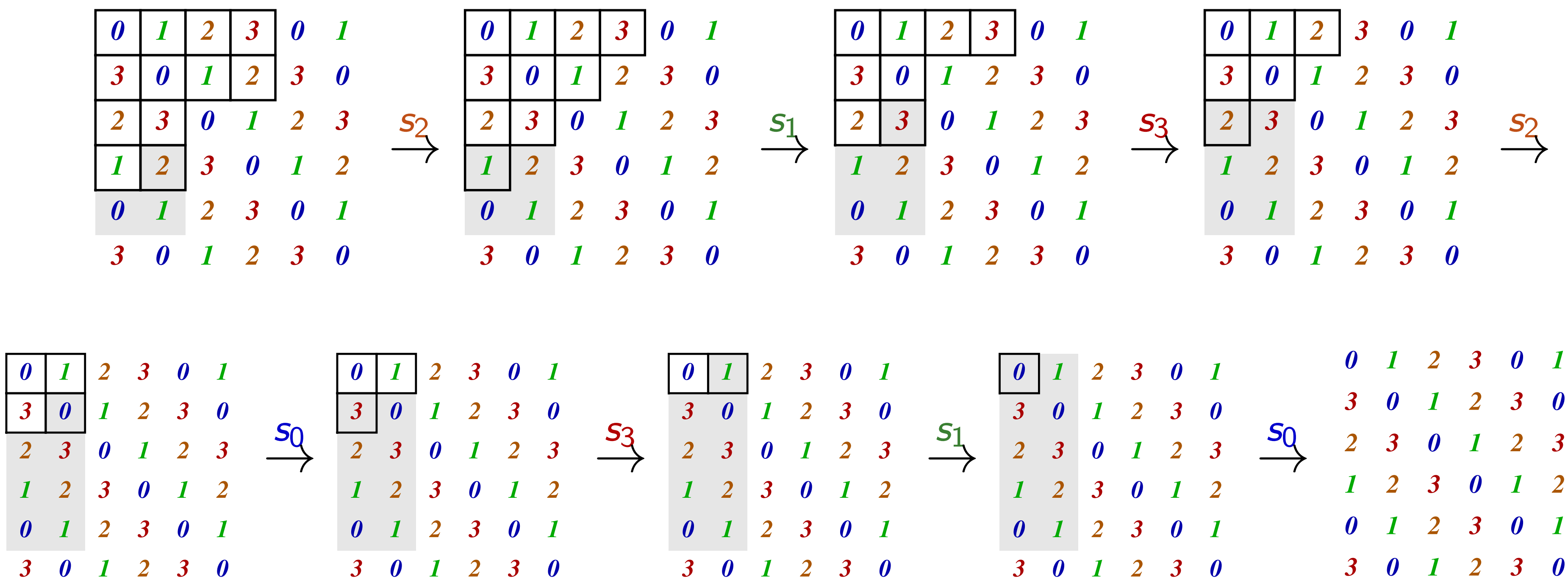
Idea: We can use this to figure out a *word* for λ.

Finding the word corresponding to a core partition.

Example: The word in S_4 corresponding to $\lambda = (6, 4, 4, 2, 2)$:

$s_1 s_0 s_2 s_1 s_3 s_2 s_0 s_3 s_1 s_0$

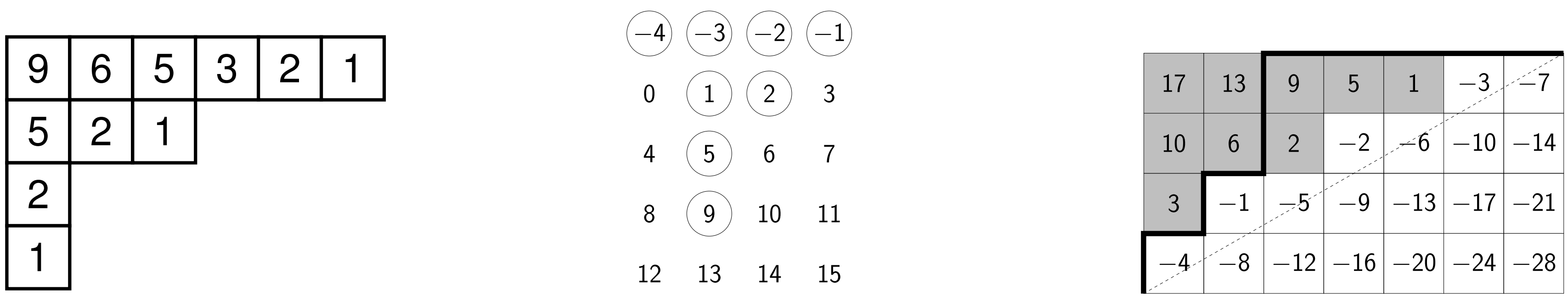




Anderson's bijection and the formula

Building on James's abacus diagrams, Anderson found a bijection:

$$\left\{ \begin{array}{c} \text{simultaneous} \\ (a,b)\text{-cores} \end{array} \right\} \overset{\textit{James}}{\longleftrightarrow} \left\{ \begin{array}{c} (a,b)\text{-flush} \\ \text{balanced abaci} \end{array} \right\} \overset{\textit{And.}}{\longleftrightarrow} \left\{ \begin{array}{c} (a,b)\text{-Dyck paths} \\ (0,0) \to (b,a) \\ \text{above } y = \frac{a}{b}x \end{array} \right\}$$



Proof that the number of (a,b)-Dyck paths is $\frac{1}{a+b}\binom{a+b}{a}$: (Bizley '55)

- Path rotation gives an equivalence relation on the set of all lattice paths from $(0,0) \to (b,a)$.
- There are $\binom{a+b}{a}$ such paths and the equivalence classes have $a+b$ elements each.

Familiar numbers

t	1	2	3	4	5	6	n
# of $(t, t+1)$-cores:	1	2	5	14	42	132	

Specialize Anderson's result:

$$\# \ (t, t+1)\text{-cores}$$
$$\frac{1}{2t+1}\binom{2t+1}{t} = \frac{1}{t+1}\binom{2t}{t}$$

Question: Is there a simple statistic on simultaneous core partitions that gives us a q-analog of the Catalan numbers?

$$\sum_{\substack{\lambda \text{ is a} \\ (t,t+1)\text{-core}}} q^{\mathrm{stat}(\lambda)} = \frac{1}{[t+1]_q} \begin{bmatrix} 2t \\ t \end{bmatrix}_q$$

Answer: **Yes.** We will create an analog of the **major statistic**.

The major statistic

For a permutation $\pi = \pi_1\pi_2\cdots\pi_n$, the **major statistic** $\mathrm{maj}(\pi)$ is the sum of the positions of the descents of π:

$$\mathrm{maj}(\pi) = \sum_{i:\pi_{i-1}>\pi_i} i.$$

For a $(t, t+1)$-core λ, create the sequence $b = (b_0, \ldots, b_{t-1})$, where $b_i = \#\ 1^{\text{st}}$ col. boxes with hook length $\equiv i \bmod t$.

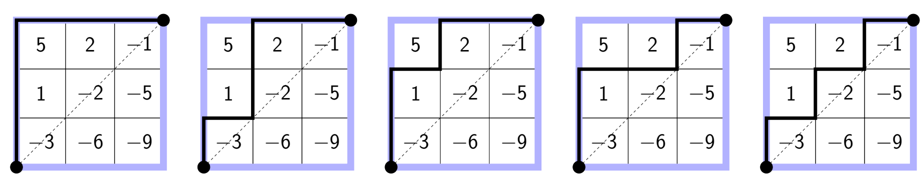
Define
$$\mathrm{maj}(\lambda) = \sum_{i\,:\,b_{i-1}\geq b_i} (2i - b_i).$$

Theorem. (AHJ '13)

$$\sum_{\substack{\lambda \text{ is a} \\ (t,t+1)\text{-core}}} q^{\mathrm{maj}(\lambda)} = \frac{1}{[t+1]_q}\begin{bmatrix} 2t \\ t \end{bmatrix}_q$$

See: maj defined as a sum over descents in a sequence.

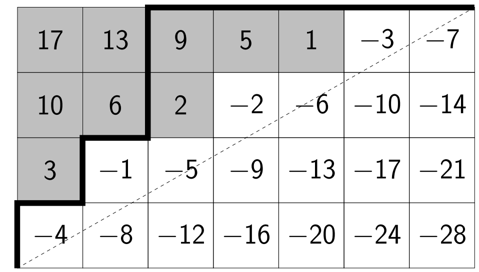
Why? Major index on Dyck paths!



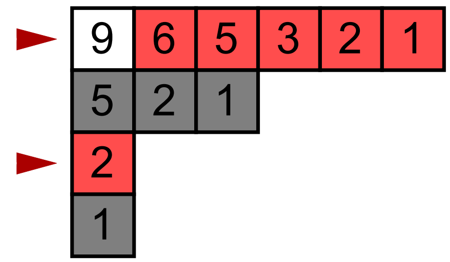
Add positions of valleys: $\frac{1}{[4]_q}\begin{bmatrix} 6 \\ 3 \end{bmatrix}_q = q^0 + q^2 + q^3 + q^4 + q^{2+4}$

The Zeta Map (via cores)

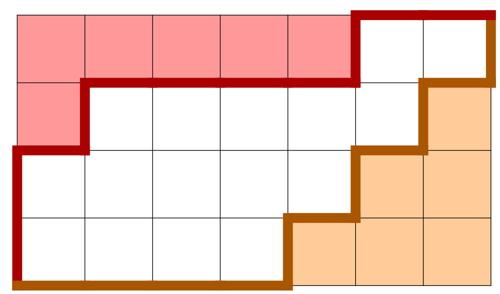
Follow this recipe:



1. Start with any (a, b)-Dyck path P.
2. Find the corresponding (a, b)-core κ.



3. Highlight the boxes
   in the a-rows and b-bdry of κ.
   (Or in b-rows and a-bdry of κ^c)
4. Let λ (μ) be the partition with those number of boxes.

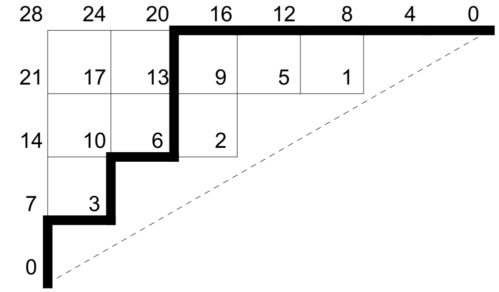


5. Draw the (a, b)-Dyck path Q (R) that bounds λ. (μ)

This defines the zeta map; $\zeta(P) = Q.$ (Or the eta map $\eta(P) = R.$)

The Zeta Map (via the sweep map)

Follow this recipe:



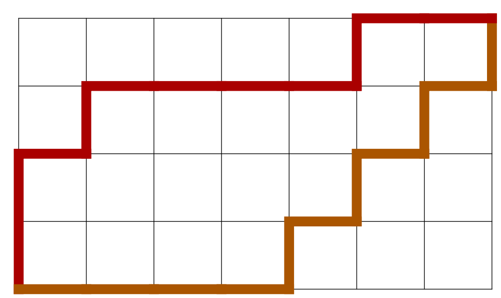
1. Start with any (a, b)-Dyck path P.
2. Assign to each lattice point its level.
3. Write down the sequence of levels $(0,0) \rightsquigarrow (b,a)$ $(b,a) \rightsquigarrow (0,0)$ with their associated N or E.
4. Sort these from smallest to largest, permuting the N's and E's too.
5. Read the steps as a new (a, b)-Dyck path Q. (R after rotating 180°.)

N	E	N	E	N	N	E	E	E	E	E
0,	7,	3,	10,	6,	13,	20,	16,	12,	8,	4

E	E	E	E	E	N	N	E	N	E	N
0,	4,	8,	12,	16,	20,	13,	6,	10,	3,	7

N	N	E	N	E	E	E	E	N	E	E
0,	3,	4,	6,	7,	8,	10,	12,	13,	16,	20

E	E	E	E	N	E	N	E	N	E	N
0,	3,	4,	6,	7,	8,	10,	12,	13,	16,	20



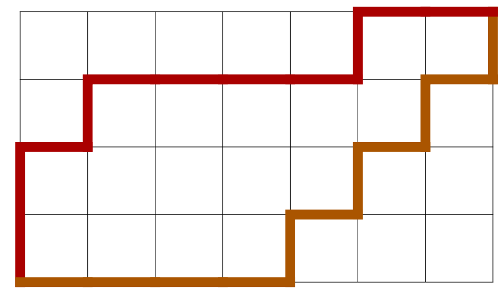
This **is also** the zeta map; $\zeta(P) = Q$! (Or the eta map $\eta(P) = R$!)

Bijection?!?!?

- What a curious rule!
- Is it even well-defined?
- Claim: ζ is a bijection!
  - Computer evidence points to yes!
  - Inverse exists for $(a, a+1)$-cores (Dyck paths!)
  - Inverse exists for $(a, am+1)$-cores
- **NEW!** If we know both Q and R, we can recover P.
- **NEW!** With a new statistic $\delta(P)$, we can iteratively recover P.

An inverse knowing $\zeta(P)$ and $\eta(P)$

1. Start with paths $\zeta(P) = Q$ above diag. and $\eta(P) = R$ rotated below diag.



2. Label steps with $1 \to a + b$.

$$\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 5 & 7 & 1 & 9 & 2 & 3 & 4 & 6 & 11 & 8 & 10 \end{array}$$

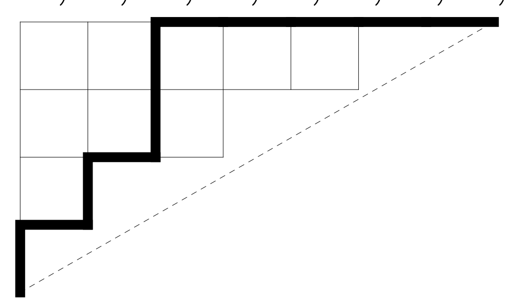
3. Read off cycle permutation γ: $\gamma(i)$ is the step in R opposite step i in Q.

$$(1, 5, 2, 7, 4, 9, 11, 10, 8, 6, 3)$$

$$\nearrow, \searrow, \nearrow, \searrow, \nearrow, \nearrow, \searrow, \searrow, \searrow, \searrow, \searrow$$

$$N, E, N, E, N, N, E, E, E, E, E$$

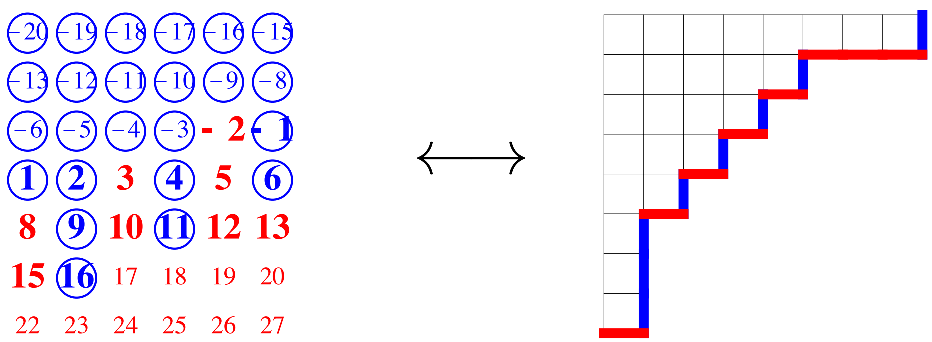
4. Recover P from γ by converting ascents to N and descents to E.



Research Questions

★ Can we extend combinatorial interps to other reflection groups?

- Yes! Involves self-conjugate partitions.
- Article (28 pp) published in *Journal of Algebra.* (2012)
  Sets up the theory.
- Article (16 pp) published in *European Journal of Comb.* (2014)
  Applies the theory.
- Joint with Brant Jones, JMU, Drew Armstrong, Miami.

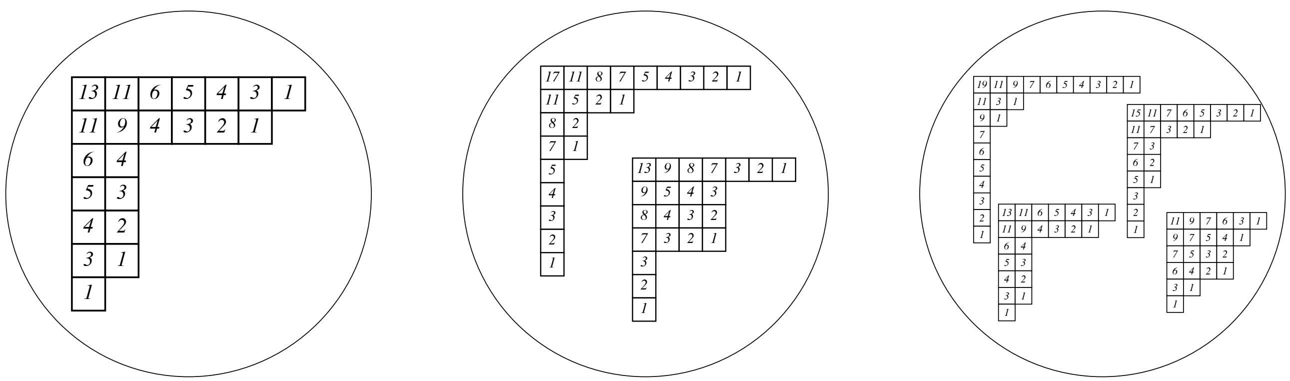


Research Questions

★ What numerical properties do self-conjugate core partitions have?

- There are more (s.c. $t+2$-cores of n) than (s.c. t-cores of n).
- Article (17 pp) published in *Journal of Number Theory.* (2013)
- Joint with Rishi Nath, York College, CUNY.

4–cores of 22 6–cores of 22 8–cores of 22



Research Questions

★ Properties of simultaneous core partitions.

- **Question:** Is there a core statistic for a q-analog of $\frac{1}{s+t}\binom{s+t}{s}$?
- *Progress:* m-Catalan number C_3 through $(3, 3m+1)$-cores.
- **Question:** How do we find the statistic $\delta(P)$ from path $\zeta(P)$?
- *Progress:* Known in certain cases.
- Article (34 pp) to appear in *J. Combinatorial Theory Ser. A.*

- **Question:** Why is the zeta map a bijection?
- *Progress:* Mystère et boule de gomme!

★ Happy to have students who would like to do research! ★