

Course Notes

Combinatorics, Fall 2018

Queens College, Math 636

Prof. Christopher Hanusa

<http://qcpages.qc.cuny.edu/~chanusa/courses/636/18/>

Reference List

The following are books that I recommend to complement this course. They are *on reserve* in the library.

Benjamin and Quinn. *Proofs that really count*.

Bóna. *A walk through combinatorics*.

Brualdi. *Introductory combinatorics*.

Graham, Knuth, and Patashnik. *Concrete mathematics*.

Mazur. *Combinatorics: A guided tour*

van Lint and Wilson. *A course in combinatorics*.

What is combinatorics?

In this class: Learn how to count . . .

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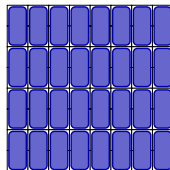
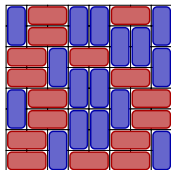
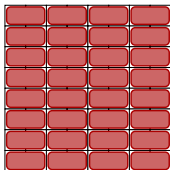
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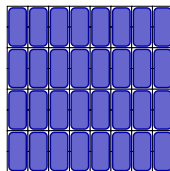
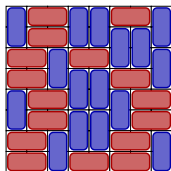
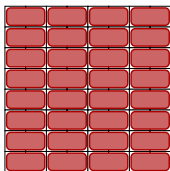
A **domino tiling** is a placement of dominoes on a region, where

- ▶ Each domino covers two squares.
- ▶ The dominoes cover the whole region and do not overlap.

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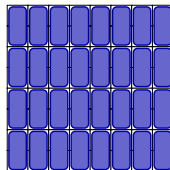
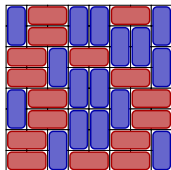
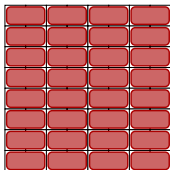
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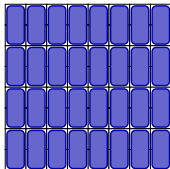
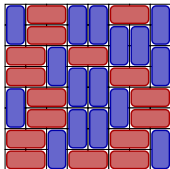
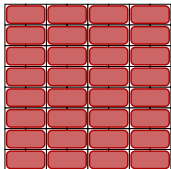
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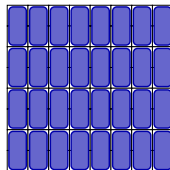
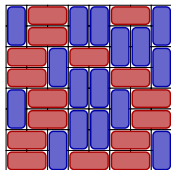
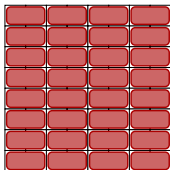
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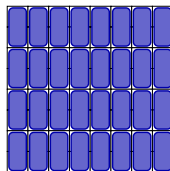
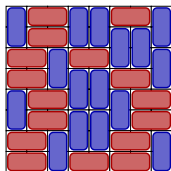
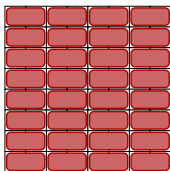
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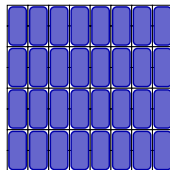
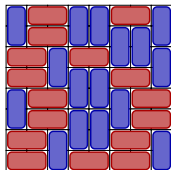
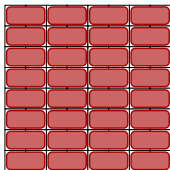
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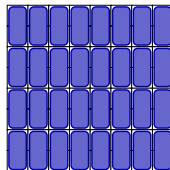
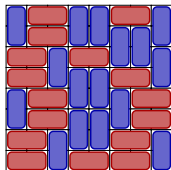
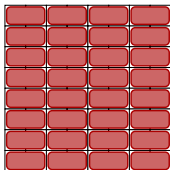
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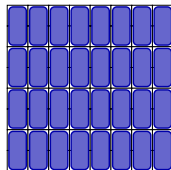
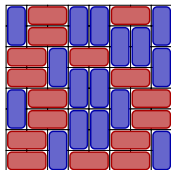
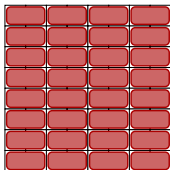
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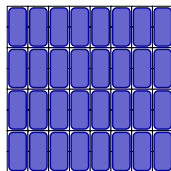
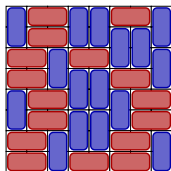
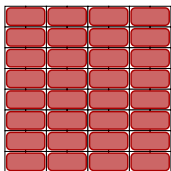
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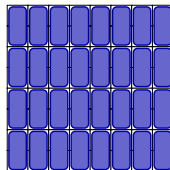
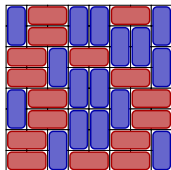
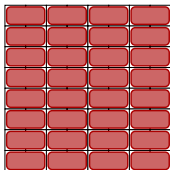
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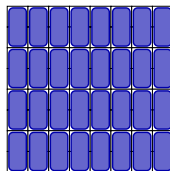
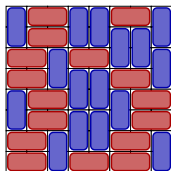
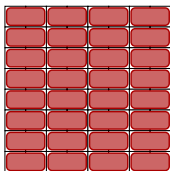


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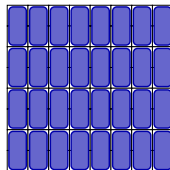
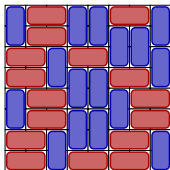
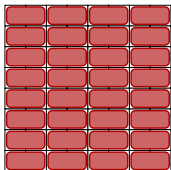
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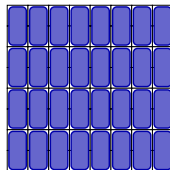
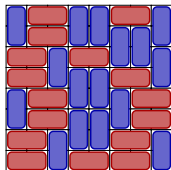
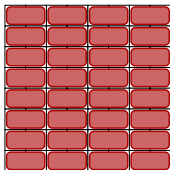
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Domino tilings

How to determine the “answer”?

- ▶ Convert the chessboard into a combinatorial structure (a graph).
- ▶ Represent the graph numerically as a matrix.
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Answer: If m and n are both even, then we have the **formula** (!):

$$\prod_{j=1}^{m/2} \prod_{k=1}^{n/2} \left(4 \cos^2 \frac{\pi j}{m+1} + 4 \cos^2 \frac{\pi k}{n+1} \right).$$

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Combinatorial questions

Given some discrete objects, what properties and structures do they have?

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Mastering “Combinatorics” means internalizing many different techniques and strategies to know the best way to approach any counting question. We will develop **our toolbox**.

Uses a different kind of reasoning than in other math classes.

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Visit the webpage. First homework (many parts!) due Wed.

Numbers are everywhere

Arrange yourselves into groups of **four** people,
With people you **don't know**.

- ▶ Introduce yourself. (your name, where you are from)
- ▶ What brought you to this class?
- ▶ Fill out **the front of** your notecard:
 - ▶ Write your name. (Stylize if you wish.)
 - ▶ Write some words about how I might remember you & your name.
 - ▶ *Draw* something (anything!) in the remaining space.
- ▶ Exchange contact information. (phone / email / other)
- ▶ *Small talk suggestion:* What kept you busy this summer?

Four Counting Questions (p. 2)

Here are four counting questions.

- Q1. How many 8-character passwords are there using $A-Z$, $a-z$, $0-9$?
- Q2. In how many ways can a baseball manager order nine fixed baseball players in a lineup?
- Q3. How many Pick-6 lottery tickets are there?
(Choose six numbers between 1–40.)
- Q4. How many possible orders for a dozen donuts are there when the store has 30 varieties?

Group discussion: Use your powers of estimation to order these from smallest to largest.

Counting words

Definition: A **list** or **word** is an ordered sequence of objects.

Definition: A **k -list** or **k -word** is a list of length k .

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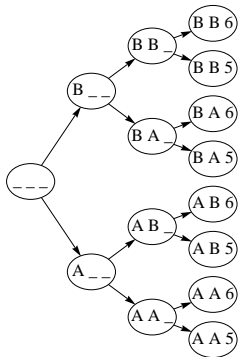
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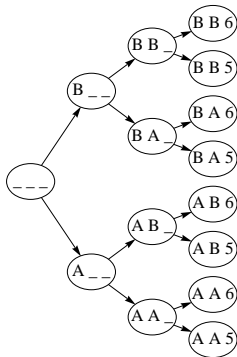
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Alternatively: Notice two *independent* choices for each character. Multiply $2 \cdot 2 \cdot 2 = 8$.



The Product Principle

This illustrates:

The product principle: When counting lists (l_1, l_2, \dots, l_k) ,

IF there are c_1 choices for entry l_1 , each leading to a different list,

AND IF there are c_i choices for entry l_i ,

no matter the choices made for l_1 through l_{i-1} ,
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THEN there are $c_1 c_2 \cdots c_k$ such lists.

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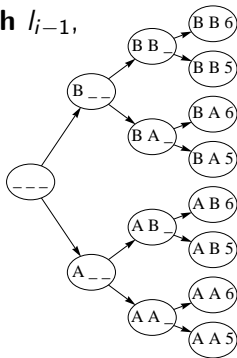
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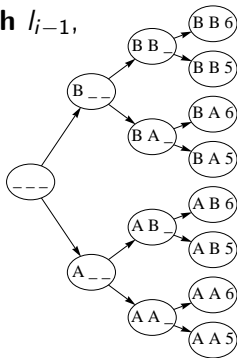
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Caution: The product principle seems simple, but we must be careful when we use it.



Lists WITH repetition

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Answer: Creating a word of length 8, with ____ choices for each character. Therefore, the number of 8-character passwords is ____.
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In general, the number of words of length k that can be made from an alphabet of length n and where repetition is allowed is n^k

Application: Counting Subsets

Example. How many subsets of a set $S = \{s_1, s_2, \dots, s_n\}$ are there?

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Equiv.: We can label the subsets by whether or not they contain s_i .

For example, for $n = 3$, we label the subsets $\left\{ \begin{array}{l} 000, 100, 010, 110, \\ 001, 101, 011, 111 \end{array} \right\}$.

Permutations

Q2. In how many ways can a baseball manager order nine fixed baseball players in a lineup?

Answer: The number of choices for each lineup spot are:

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Question: How many k -permutations of n are there?

Lists WITHOUT repetition

Question: How many 8-character passwords are there using $A-Z$, $a-z$, $0-9$, containing no repeated character?

OK: 2eas3FGS, 10293465

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In general, the number of words of length k that can be made from an alphabet of length n and where repetition is NOT allowed is $(n)_k$.

- ▶ That is, the number of k -permutations of an n -set is $(n)_k$.
- ▶ **Special case:** For n -permutations of an n -set: $n!$.

Notation

Some quantities appear frequently, so we use shorthand notation:

▶ $[n] := \{1, 2, \dots, n\}$ ▶ $2^S :=$ set of all subsets of S

▶ $n! := n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$

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▶ $\binom{n}{k} := \frac{n!}{k!(n-k)!} = \frac{(n)_k}{k!}$

▶ $\binom{\binom{n}{k}}{k} := \binom{k+n-1}{k}$

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Answer: $\binom{40}{6} = 3,838,380$.

- ▶ $\binom{n}{k}$ is called a **binomial coefficient**.
- ▶ Alternate phrasing: How many k -subsets of an n -set are there?
- ▶ The individual objects we are counting are unordered. They are subsets, not lists.

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Let's rearrange it.

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Since we counted the same quantity twice, they must be equal!

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Think Write Pair Share: Enumerate **all** multisubsets of $[3]$.
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Answer:

How would you describe a k -multisubset of $[n]$?

Stars and Bars

Question: How many k -multisets
can be made from an n -set?

— *is the same as* —

Question: How many ways are there
to place k indistinguishable balls
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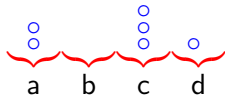
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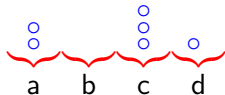
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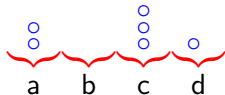
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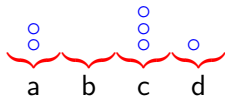
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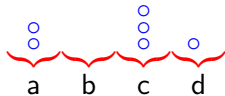
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$$\binom{k+n-1}{k} =: \binom{n}{k}$$

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- Q4. How many possible orders for a dozen donuts are there when the store has 30 varieties?

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Q4. How many possible orders for a dozen donuts are there when the store has 30 varieties?

Answer: $\binom{30}{12} = \binom{30}{18} =$

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Correct order:

Q2. Order 9 baseball players	$(9!)$	362,880
Q3. Pick-6; numbers 1–40	$\binom{40}{6}$	3,838,380
Q4. 12 donuts from 30	$\binom{30}{12}$	7,898,654,920
Q1. 8-character passwords	(62^8)	218,340,105,584,896

Summary

	order matters (choose a list)	order doesn't matter (choose a set)
repetition allowed		
repetition not allowed		

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