Introduction to Bijections

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- Proving it is a bijection (requires logical reasoning).

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Example. Let A be the set of 3-subsets of $[n]$ and let B be the set of 3-lists of $[n]$. Then define $f : A \to B$ to be the function that takes a 3-subset $\{i_1, i_2, i_3\} \in A$ (with $i_1 \leq i_2 \leq i_3$) to the word $i_1 i_2 i_3 \in B$.

Question: Is $\mathrm{rng}(f) = B$?

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What is an example of a function that is onto and not one-to-one?

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Step 1: Find a candidate bijection.

Strategy. Try out a small (enough) example. Try $n = 5$ and $k = 2$.

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Guess: Let S be a k-subset of $[n]$. Perhaps $f(S) =$ ______.

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Step 2: Prove f is well defined.

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f is 1-to-1:

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We conclude that f is a bijection and therefore, $\binom{n}{k} = \binom{n}{n-k}$.

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Prove that a rule f is a bijection by finding f's **inverse**:

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Then both f and g are bijections.

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Proof. Let A be the set of even-sized subsets of $[n]$ and let B be the set of odd-sized subsets of $[n]$. Consider the function

$$f(S) = \begin{cases} S \setminus \{1\} & \text{if } 1 \in S \\ S \cup \{1\} & \text{if } 1 \notin S \end{cases}.$$

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Eyebrow-Raising Consequence: $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.