What is a Combinatorial Proof?

Definition: A **combinatorial interpretation** of a numerical quantity is a set of combinatorial objects that is counted by the quantity.

Example. We can choose k objects out of n total objects in $\binom{n}{k}$ ways.

Use this fact "backwards" by interpreting an occurrence of $\binom{n}{k}$ as the number of ways to choose k objects out of n.

This leads to my favorite kind of proof:

Definition: A **combinatorial proof** of an identity $X = Y$ is a proof by counting (!). You find a set of objects that can be interpreted as a combinatorial interpretation of both the left hand side (LHS) and the right hand side (RHS) of the equation. As both sides of the equation count the same set of objects, they must be equal!

- It is important to get the set of objects right.
- To do this, you must ask a good question: "In how many ways..."

A Simple Combinatorial Proof

Example. Prove *Equation* (2.2): For $0 \leq k \leq n$, $\binom{n}{k} = \binom{n}{n-k}$.
(We already know a bijective proof of this fact.)

Analytic Proof: $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!} = \binom{n}{n-k}$

Combinatorial Proof:

Question: In how many ways can we adopt k of n cats available for adoption at the animal shelter?

Answer 1: Choose k of the n cats to adopt in $\binom{n}{k}$ ways.

Answer 2: Choose $n-k$ of the n cats to NOT adopt in $\binom{n}{n-k}$ ways.

Because the two quantities count the same set of objects in two different ways, the two answers are equal. $\square$

Another Simple Combinatorial Proof

Example. Prove *Equation* (2.4): $k\binom{n}{k} = n\binom{n-1}{k-1}$.

Analytic Proof:

Combinatorial Proof:

Question: In how many ways can we choose from n club members a committee of k members with a chairperson?

Answer 1:

Answer 2:

Because the two quantities count the same set of objects in two different ways, the two answers are equal. □

Pascal's Identity

Example. Prove *Theorem* 2.2.1: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Combinatorial Proof:

Question: In how many ways can we choose k flavors of ice cream if n different choices are available?

Answer 1:

Answer 2:

Because the two quantities count the same set of objects in two different ways, the two answers are equal. □

Summing Binomial Coefficients

Example. Prove *Equation* (2.3): $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$.

Analytic Proof: ???

Combinatorial Proof:

Question: How many subsets of $\{1, 2, \ldots, n\}$ are there?

Answer 1: Condition on how many elements are in a subset.

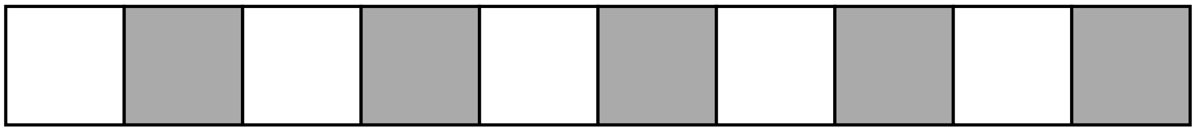
Answer 2:

Because the two quantities count the same set of objects in two different ways, the two answers are equal. □

—Worksheet—

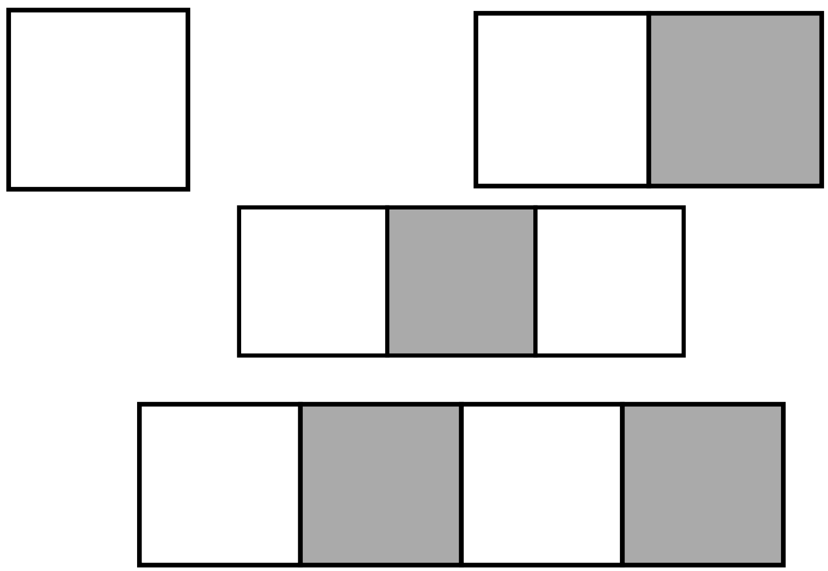
Tiling a board with dominos and squares

Question: How many ways are there to tile a $1 \times n$ board using only dominoes and squares?



Definition: Let $f_n = \#$ of ways to tile a $1 \times n$ board.

$f_0 = 1$
$f_1 =$
$f_2 =$
$f_3 =$
$f_4 =$



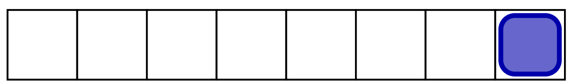
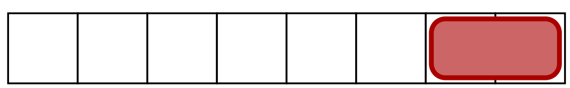
Why Fibonacci?

Fibonacci numbers f_n satisfy

- $f_0 = f_1 = 1$ ✓
- $f_n = f_{n-1} + f_{n-2}$ ✓

There are f_n tilings of a $1 \times n$ board

Every tiling ends in either:

- a square 
  - **How many?** Fill the initial $1 \times (n-1)$ board in f_{n-1} ways.
- a domino 
  - **How many?** Fill the initial $1 \times (n-2)$ board in f_{n-2} ways.

Total: $f_{n-1} + f_{n-2}$

Fibonacci identities

We have a new definition for Fibonacci:

$$f_n = \text{the number of square-domino tilings of a } 1 \times n \text{ board.}$$

This *combinatorial interpretation* of the Fibonacci numbers provides a framework to prove identities.

- Did you know that $f_{2n} = (f_n)^2 + (f_{n-1})^2$?

f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}
1	2	3	5	8	**13**	**21**	34	55	89	144	233	377	**610**

$$f_{14} = f_7^2 + f_6^2$$

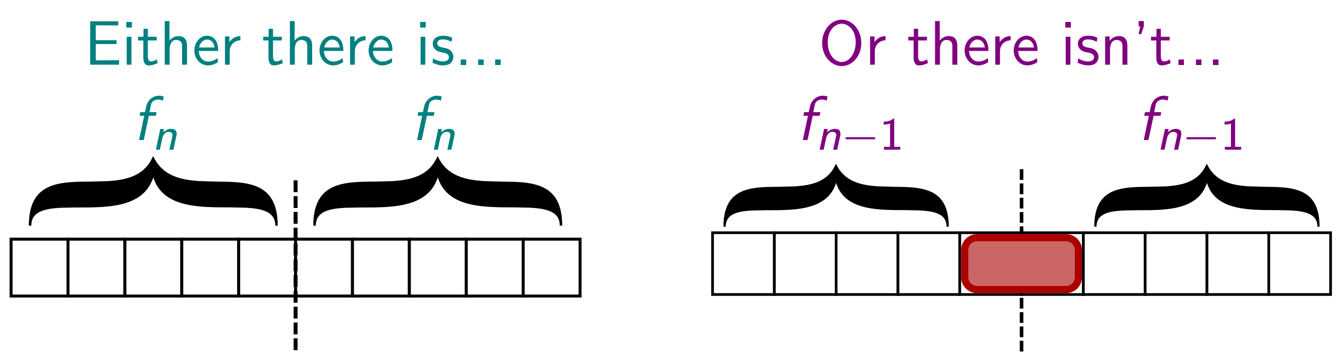
$$610 = 441 + 169$$

Proof that $f_{2n} = (f_n)^2 + (f_{n-1})^2$

Proof. How many ways are there to tile a $1 \times (2n)$ board?

Answer 1. Duh, f_{2n}.

Answer 2. Ask whether there is a break in the middle of the tiling:



For a total of $(f_n)^2 + (f_{n-1})^2$ tilings.

We counted f_{2n} in two different ways, so they must be equal. $\square$

Further reading:

Arthur T. Benjamin and Jennifer J. Quinn
Proofs that Really Count, MAA Press, 2003.