# mmooRREE COUNTING!

*Question:* In how many ways can we place $k$ objects in $n$ boxes?

*Answer:* It depends.

- What do the objects look like?
  - Do the objects all look the same?
- What do the boxes look like?
  - Do the boxes all look the same?
- Are there any restrictions?
  - Is there a size limit?
  - Must there be an object in each box?

# Counting distributions

*Definition:* A **distribution** is an assignment of objects to recipients.

Certain counting problems can be revisited in this framework:

$$\left\{ \begin{array}{c} \text{Five-letter passwords} \\ \text{on } \{A, B, C, D, E, F, G\} \\ \end{array} \right\} \text{ correspond to } \left\{ \begin{array}{c} \text{Distributions of} \\ \_\_\_ \text{ distinct objects} \\ \text{into } \_\_\_ \text{ distinct boxes} \\ \end{array} \right\}$$

- What are candidates for objects, boxes?
- View as a function
- View as a distribution
- Find the restriction

# THE CHART

*Question:* In how many ways can we place $k$ objects in $n$ boxes?

| **Distributions of** | | Restrictions on # objects received | | | |
|---|---|---|---|---|---|
| $k$ **objects** | $n$ **boxes** | none | $\leq 1$ | $\geq 1$ | $= 1$ |
| distinct | distinct | | | | |
| identical | distinct | | | | |
| distinct | identical | | | | |
| identical | identical | | | | |

Where do our known answers fit into the table? (Use function view)

- $n^k$: Objects distinct, Boxes distinct, no restriction.
- $(n)_k$: Objects distinct, Boxes distinct, $\leq 1$ object per box.
- $n!$: Permutations. What about when $n \neq k$?
- $\binom{n}{k}$: Objects ________, Boxes ________, ________________.
- $\left(\!\binom{n}{k}\!\right)$: Objects ________, Boxes ________, ________________.

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|---|---|---|---|---|---|
| $k$ **objects** | $n$ **boxes** | none | $\leq 1$ | $\geq 1$ | $= 1$ |
| distinct | distinct | | | | |
| identical | distinct | | | | |
| distinct | identical | | | | |
| identical | identical | | | | |

We can also fill in these answers:

- Objects identical, Boxes distinct, $\geq 1$ object per box:
- Objects identical, Boxes distinct, $= 1$ object per box:

# Distinct objects in indistinguishable boxes

When placing $k$ distinguishable objects into $n$ indistinguishable boxes, what matters? ____________________________

- Each object needs to be in some box.
- No object is in two boxes.

We have rediscovered ______________.

So ask "How many set partitions are there of a set with $k$ objects?"

Or even, "How many set partitions are there of $k$ objects into $n$ parts?"

# Stirling numbers

The **Stirling number of the second kind** counts the number of ways to partition a set of $k$ elements into $i$ **non-empty** subsets.

Notation: $S(k,i)$ or $\left\{ {k \atop i} \right\}$. ← **Careful about this order!**

| $k$ | $\left\{ {k \atop 0} \right\}$ | $\left\{ {k \atop 1} \right\}$ | $\left\{ {k \atop 2} \right\}$ | $\left\{ {k \atop 3} \right\}$ | $\left\{ {k \atop 4} \right\}$ | $\left\{ {k \atop 5} \right\}$ | $\left\{ {k \atop 6} \right\}$ | $\left\{ {k \atop 7} \right\}$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | |
| 1 | | 1 | | | | | | |
| 2 | | 1 | 1 | | | | | |
| 3 | | 1 | 3 | 1 | | | | |
| 4 | | 1 | 7 | 6 | 1 | | | |
| 5 | | 1 | 15 | 25 | 10 | 1 | | |
| 6 | | 1 | 31 | 90 | 65 | 15 | 1 | |
| 7 | | 1 | | | | | | 1 |

In Stirling's triangle:

$S(k,1) = S(k,k) = 1.$

$S(k,2) = 2^{k-1} - 1.$

$S(k,k-1) = \binom{k}{2}.$

Later: Formula for $S(k,i)$.

To fill in the table, find a recurrence for $S(k,i)$:

**Ask:** In how many ways can we place $k$ objects into $i$ boxes?

We'll condition on the placement of element $\#i$:

# THE CHART

*Question:* In how many ways can we place $k$ objects in $n$ boxes?

| **Distributions of** | | Restrictions on # objects received | | | |
|---|---|---|---|---|---|
| $k$ **objects** | $n$ **boxes** | none | $\leq 1$ | $\geq 1$ | $= 1$ |
| distinct | distinct | $n^k$ | $(n)_k$ | | $n!$ or 0 |
| identical | distinct | $\left(\!\binom{n}{k}\!\right)$ | $\binom{n}{k}$ | $\left(\!\binom{n}{k-n}\!\right)$ | 1 or 0 |
| distinct | identical | | | | |
| identical | identical | | | | |

$S(k, n)$ counts ways to place $k$ distinct obj. into $n$ identical boxes.

What if we then label the boxes?
(*Note that here we have counted onto functions* $[k] \to [n]$.)

How many ways to distribute distinct objects into identical boxes:

- If there is exactly one item in each box?
- If there is at most one item in each box?
- What about with no restrictions? ($n \geq k \rightsquigarrow$ Bell number $B_k$)

# Bell numbers

*Definition:* The **Bell number** $B_k$ is the number of partitions of a set with $k$ elements, into any number of non-empty parts.

We have $B_k = S(k,0) + S(k,1) + S(k,2) + \cdots + S(k,k)$.

| $B_0$ | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ | $B_6$ | $B_7$ | $B_8$ | $B_9$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 5 | 15 | 52 | 203 | 877 | 4140 | 21147 |

*Theorem 2.3.3.* The Bell numbers satisfy a recurrence:

$$B_k = \binom{k-1}{0} B_0 + \binom{k-1}{1} B_1 + \cdots + \binom{k-1}{k-1} B_{k-1}.$$

*Proof:* How many partitions of $\{1, \ldots, k\}$ are there?

LHS: $B_k$, obviously.

RHS: Condition on the box containing the last element $k$:
How many partitions of $[k]$ contain $i$ elements in the box with $k$?

# Indistinguishable objects in indistinguishable boxes

When placing $k$ indistinguishable objects into $n$ indistinguishable boxes, what matters? ______________________

- We are partitioning the **integer** $k$ instead of the **set** $[k]$.

Example. What are the partitions of 6?

*Definition:* $P(k, i)$ is the number of partitions of $k$ into $i$ parts.

Example. We saw $P(6, 1) = 1$, $P(6, 2) = 3$, $P(6, 3) = 3$, $P(6, 4) = 2$, $P(6, 5) = 1$, and $P(6, 6) = 1$.

*Definition:* $P(k)$ is the number of partitions of $k$ into any number of parts.

Example. $P(6) = 1 + 3 + 3 + 2 + 1 + 1 = 11.$

# THE CHART, COMPLETED

*Question:* In how many ways can we place $k$ objects in $n$ boxes?

| **Distributions of** | | Restrictions on # objects received | | | |
|---|---|---|---|---|---|
| $k$ **objects** | $n$ **boxes** | none | $\leq 1$ | $\geq 1$ | $= 1$ |
| distinct | distinct | $n^k$ | $(n)_k$ | $n!S(k,n)$ | $n!$ or 0 |
| identical | distinct | $\left(\binom{n}{k}\right)$ | $\binom{n}{k}$ | $\left(\binom{n}{k-n}\right)$ | 1 or 0 |
| distinct | identical | $\sum S(k,i)$ | 1 or 0 | $S(k,n)$ | 1 or 0 |
| identical | identical | $\sum P(k,i)$ | 1 or 0 | $P(k,n)$ | 1 or 0 |

$P(k,n)$ counts ways to place $k$ identical obj. into $n$ identical boxes.

How many ways to distribute identical objects into identical boxes:

- If there is exactly one item in each box?
- If there is at most one item in each box?
- What about with no restrictions?

(This is the # of integer partitions of $k$ into at most $n$ parts.)