Principle of Inclusion-Exclusion

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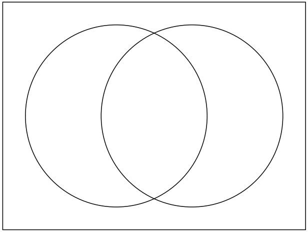
Solution.

Let S be the set of students who play soccer and B be the set of students who play basketball. Then, $|S \cup B| = |S| + |B|$ ________.



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$$\begin{aligned} |A_1 \cup A_2| &= |A_1| + |A_2| \qquad\qquad - |A_1 \cap A_2| \\ |A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| \\ &\qquad - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| \end{aligned}$$

$$|A_1 \cup \cdots \cup A_m| = \sum |A_i| - \sum |A_i \cap A_j| + \sum \left|A_i \cap A_j \cap A_k\right| \cdots$$

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It may be more convenient to apply inclusion/exclusion where the A_i are *forbidden* subsets of $\mathcal{U}$, in which case ______________.

mmm. . .PIE

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Notation: $\pi = p_1 p_2 \cdots p_n$ is the one-line notation for a permutation of [n] whose first element is p_1, second element is p_2, etc.

Example. How many permutations $p = p_1 p_2 \cdots p_n$ are there in which at least one of p_1 and p_2 are even?

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Applying PIE: So $|A_1 \cup A_2| =$

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Let $A_1 \subset \mathcal{U}$ be the multiples of 5, $A_2 \subset \mathcal{U}$ be the multiples of 6, and $A_3 \subset \mathcal{U}$ be the multiples of 8. We want $|\mathcal{U}| - |A_1 \cup A_2 \cup A_3|$.

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Combinations with Repetitions

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What we would like to calculate is:

> In how many ways can we choose k elements out of an arbitrary multiset?

Now, it's as easy as PIE.

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Now calculate: $|\mathcal{U}| =$ $|A_1| =$ $|A_2| = \left(\!\binom{3}{5}\!\right)$ $|A_3| = \left(\!\binom{3}{4}\!\right)$
$|A_1 \cap A_2| = 3$ $|A_1 \cap A_3| = 1$ $|A_2 \cap A_3| = 0$ $|A_1 \cap A_2 \cap A_3| = 0$

And finally: So $|\mathcal{U}| - |A_1 \cup A_2 \cup A_3| =$

Derangements

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Definition: An n-derangement is an n-permutation $\pi = p_1 p_2 \cdots p_n$ such that $p_1 \neq 1$, $p_2 \neq 2$, $\cdots$, $p_n \neq n$.

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Notation: We let D_n be the number of all n-derangements.

When you see D_n, think combinatorially: "The number of ways to return n hats to n people so no one gets his/her own hat back"

Calculating the number of derangements

Example. Calculate D_n.

Solution. Let $\mathcal{U}$ be the set of all n-permutations.
Remove bad permutations using PIE.
For all i from 1 to n, define A_i to be n-perms where $p_i = i$.

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Now calculate: $|\mathcal{U}| =$ $|A_1| =$ $|A_2| =$
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When intersecting k sets, $|A_{i_1} \cap \cdots \cap A_{i_k}| =$

Recall: $|A_1 \cup \cdots \cup A_n| = \sum |A_i| - \sum |A_i \cap A_j| + \sum \left|A_i \cap A_j \cap A_k\right| \cdots$

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Therefore, $D_n = |\mathcal{U}| - |A_1 \cup \cdots \cup A_n| =$

Randomly returning hats

Upon simplification, we see

$$\begin{aligned} D_n &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n \binom{n}{n} 0! \\ &= n! - \frac{n!}{1!} + \frac{n!}{2!} - \cdots + (-1)^n \frac{n!}{n!} \\ &= n! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right] \end{aligned}$$

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$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

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Plug in $x = -1$ and truncate after n terms to see that

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Conclusion: If n people go to a party and the hats are passed back randomly, the probability that no one gets his or her hat back at the party is $D_n/n!$, which is approximately $1/e \approx 37\%$.

Combinatorial proof involving D_n

Recall: The combinatorial interpretation of D_n is: "The number of ways to return n hats to n people so no one gets his/her own hat back"

Example. Prove the following recurrence relation for D_n combinatorially.

$$D_n = (n-1)\big(D_{n-2} + D_{n-1}\big)$$

A formula for Stirling numbers (p. 90)

Recall: $S(n,k) = \left\{ {n \atop k} \right\}$ is the number of **set partitions** of $[n]$ into exactly k parts

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Question: What is a formula for $S(n,k)$?

Solution. We will find the number of surjections from $[n]$ to $[k]$.
Use PIE with $\mathcal{U}$ = set of all functions from $[n]$ to $[k]$.
We will remove the "bad" functions where the range is not $[k]$.

A formula for Stirling numbers (p. 90)

(Careful: change of notation !!)

Recall: $S(n,k) = \left\{ {n \atop k} \right\}$ is the number of **set partitions** of $[n]$ into exactly k parts, and $k!S(n,k)$ is the number of onto functions $[n]\to[k]$.

Question: What is a formula for $S(n,k)$?

Solution. We will find the number of surjections from $[n]$ to $[k]$.
Use PIE with $\mathcal{U}$ = set of all functions from $[n]$ to $[k]$.
We will remove the "bad" functions where the range is not $[k]$.

Define A_i be the set of functions $f : [n] \to [k]$ where i is not "hit".

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In words, $A_{i_1} \cap \cdots \cap A_{i_j}$ are functions where none of i_1 through i_j are elements of the image.

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We calculate: $|\mathcal{U}| = k^n$, $|A_i| = (k-1)^n$, $|A_i \cap A_j| = (k-2)^n$
When intersecting j sets, $|A_{i_1} \cap \cdots \cap A_{i_j}| = (k-j)^n$.

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Therefore, $k!S(n,k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$; we conclude
$S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$

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Therefore, $k!S(n,k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$; we conclude
$S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n$.

A formula for Bell numbers (p. 166)

(Careful: change of notation !!)

Recall: B_n is the number of partitions of $[n]$ into any number of parts.
Manipulate our expression from prev. page to find a formula.

$$B_n = \sum_{k \geq 0} \left\{ {n \atop k} \right\} = \sum_{k \geq 0} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} j^n$$

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Recall: B_n is the number of partitions of $[n]$ into any number of parts. Manipulate our expression from prev. page to find a formula.

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Theorem 4.3.3. For any $n > 0$, $B_n = \frac{1}{e} \sum_{j\geq 0} \frac{j^n}{j!}$.

A formula for Bell numbers (p. 166)

(Careful: change of notation !!)

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Theorem 4.3.3. For any $n > 0$, $B_n = \frac{1}{e} \sum_{j\geq 0} \frac{j^n}{j!}$.

For example, $B_5 = \frac{1}{e}\left(\frac{0^5}{0!} + \frac{1^5}{1!} + \frac{2^5}{2!} + \frac{3^5}{3!} + \frac{4^5}{4!} + \frac{5^5}{5!} + \cdots \right) = 52$.