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Step 2: Massage the sums to find copies of $A(x)$.
LHS: Re-index, find missing term; RHS: separate into pieces.

Conversion to functions of $A(x)$:

Solving recurrence relations

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When the degree of the numerator is smaller than the degree of the denominator, we can use *partial fractions!* to determine an expression for $A(x)$ of the form:

$$A(x) = \frac{C_1}{1-2x} + \frac{C_2}{1-x}$$

Solving gives $A(x) = \frac{1}{1-2x} + \frac{-1}{1-x}$; each of which can be expanded:

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$$A(x) = \sum_{k\geq 0} 2^k x^k + (-1)\sum_{k\geq 0} 1^k x^k = \sum_{k\geq 0}(2^k - 1)\, x^k$$

Therefore, $a_k = 2^k - 1$.

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Practicality: $(1+\sqrt{5})/2 \approx 1.61803$ and 1 mi ≈ 1.609344 km

Multiplying two generating functions (Convolution)

Let $A(x) = \sum_{k \geq 0} a_k x^k$ and $B(x) = \sum_{k \geq 0} b_k x^k$.

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Combinatorial interpretation of the convolution:

If a_k counts all "A" objects of "size" k, and
b_k counts all "B" objects of "size" k,

Then $[x^k]\big(A(x)B(x)\big)$ counts all pairs of objects (A, B) with *total* size k.

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Example. In how many ways can we fill a halloween bag w/30 candies, where for each of 20 BIG candy bars, we can choose at most one, and for each of 40 different small candies, we can choose as many as we like?

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So, $[x^k]B(x)S(x)$ counts pairs of the form $\vee$ w/k total candies.

(some number of big candies, some number of small candies)

Vandermonde's Identity (p. 117)

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j} \binom{n}{k-j}$$

Combinatorial proof

Generating function proof

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Similarly, $\big(A(x)\big)^n = \sum_{k\geq 0}\left(\sum_{i_1+i_2+\cdots+i_n=k} a_{i_1}a_{i_2}\cdots a_{i_n}\right)x^k.$

$[x^k](A(x))^n$ counts *sequences* of objects $(A_1, A_2, \ldots, A_n)$, all of type A, with a total size over all objects of k.