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In order to approach counting questions involving symmetry rigorously, we use the mathematical notion of *equivalence relation*.

Equivalence relations

Definition: An **equivalence relation** $\mathcal{E}$ on a set A satisfies the following properties:

- **Reflexive**: For all $a \in A$, $a \mathcal{E} a$.
- **Symmetric**: For all $a, a' \in A$, if $a \mathcal{E} a'$, then $a' \mathcal{E} a$.
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- *Theorem 1.4.3.* If $a\mathcal{E}a'$, then $\mathcal{E}(a) = \mathcal{E}(a')$.
- Every element of A is in *one* and *only one* equivalence class.
  - We say: "The equivalence classes of $\mathcal{E}$ partition A."

Equivalence classes partition A

Definition: A **partition** of a set S is a set of non-empty disjoint subsets of S whose union is S.

Example. Partitions of $S = \{\star, \heartsuit, \clubsuit, ?\}$ include:

- $\{\{\star, \heartsuit\}, \{?\}, \{\clubsuit\}\}$
- $\{\{\heartsuit, \clubsuit\}, \{\star, ?\}\}$

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Key idea: (Thm 1.4.5) The set of equivalence classes of A partitions A.

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The equivalence principle: (p. 37) Let $\mathcal{E}$ be an equivalence relation on a finite set A. If every equivalence class has size C, then $\mathcal{E}$ has $|A|/C$ equivalence classes. (DIVISION!)

Permutations of multisets

Example. How many different orderings are there of the letters in the word MISSISSIPPI?

Setup: If the letters were all distinguishable, we would have a permutation of 11 letters, $\{M, P, P, I, I, I, I, S, S, S, S\}$, so $|A| =$

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Alternatively, count directly.

- In how many ways can you position the S's?
- With S's placed, how many choices for the I's?
- With S's, I's placed, how many choices for the P's?
- With S's, I's, P's placed, how many choices for the M?

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Solution. The conjugacy classes correspond to ________________.

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Define lists a and a' to be equivalent if the set of pairs is the same.

[*For example,* $(3, 2, 9, 10, 1, 5, 8, 7, 4, 6) \equiv (2, 3, 9, 10, 1, 5, 6, 4, 8, 7)$.]

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Discuss: How many different 10-lists are in the same equivalence class?

Answer:

By the equivalence principle,