# What is a Combinatorial Proof?

*Definition:* A **combinatorial interpretation** of a numerical quantity is a set of combinatorial objects that is counted by the quantity.

Example. We can choose $k$ objects out of $n$ total objects in $\binom{n}{k}$ ways.

Use this fact "backwards" by interpreting an occurrence of $\binom{n}{k}$ as the number of ways to choose $k$ objects out of $n$.

This leads to my favorite kind of proof:

*Definition:* A **combinatorial proof** of an identity $X = Y$ is a proof by counting. You find **ONE** set of objects that is a combinatorial interpretation of **BOTH** the left hand side (LHS) and the right hand side (RHS) of the equation. Because both sides of the equation count the same set of objects, they must be equal!

- Finding the right set of objects is important (and difficult).

# A Simple Combinatorial Proof

Example. Prove *Equation* (2.2): For $0 \le k \le n$, $\binom{n}{k} = \binom{n}{n-k}$.
(We already know a bijective proof of this fact.)

**Analytic Proof:** $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!} = \binom{n}{n-k}$

**Combinatorial Proof:**

*Question:* In how many ways can we adopt $k$ of $n$ cats available for adoption at the animal shelter?

*Answer 1:* Choose $k$ of the $n$ cats to adopt in $\binom{n}{k}$ ways.

*Answer 2:* Choose $n - k$ of the $n$ cats to NOT adopt in $\binom{n}{n-k}$ ways.

Because the two quantities count the same set of objects in two different ways, the two answers are equal. □

# Another Simple Combinatorial Proof

Example. Prove *Equation* (2.4): $k\binom{n}{k} = n\binom{n-1}{k-1}$.

**Analytic Proof:**

**Combinatorial Proof:**

*Question:* In how many ways can we choose from $n$ club members a committee of $k$ members with a chairperson?

*Answer 1:*

*Answer 2:*

Because the two quantities count the same set of objects in two different ways, the two answers are equal. □

# Pascal's Identity

Example. Prove *Theorem* 2.2.1: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

**Combinatorial Proof:**

*Question:* In how many ways can we choose $k$ flavors of ice cream if $n$ different choices are available?

*Answer 1:*

*Answer 2:*

Because the two quantities count the same set of objects in two different ways, the two answers are equal. □

# Summing Binomial Coefficients

Example. Prove *Equation* (2.3): $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$.

**Analytic Proof:** ???

**Combinatorial Proof:**

*Question:* How many subsets of $\{1, 2, \ldots, n\}$ are there?

*Answer 1:* Condition on how many elements are in a subset.

*Answer 2:*

Because the two quantities count the same set of objects in two different ways, the two answers are equal. $\square$

# Tiling a board with dominos and squares

*Question:* How many ways are there to tile a $1 \times n$ board using only dominoes and squares?

*Definition:* Let $f_n = \#$ of ways to tile a $1 \times n$ board.

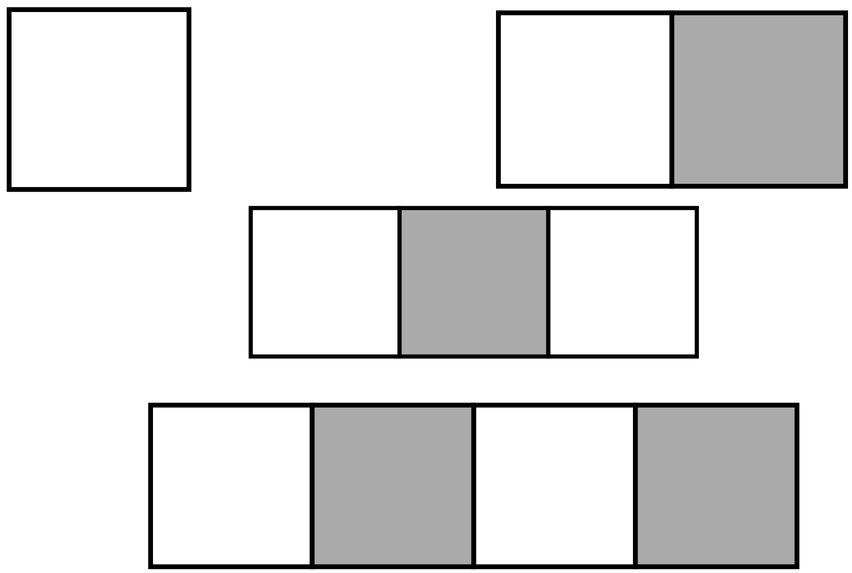
$f_0 = 1$

$f_1 =$

$f_2 =$

$f_3 =$

$f_4 =$



# Why Fibonacci?

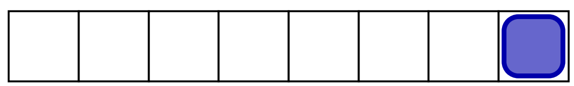
Fibonacci numbers $f_n$ satisfy

- $f_0 = f_1 = 1$ ✓
- $f_n = f_{n-1} + f_{n-2}$ ✓

There are $f_n$ tilings of a $1 \times n$ board

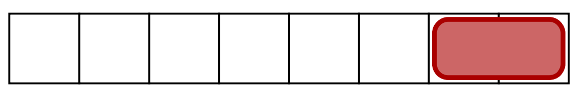
Every tiling ends in either:

- a square



  - **How many?** Fill the initial $1 \times (n-1)$ board in $f_{n-1}$ ways.

- a domino



  - **How many?** Fill the initial $1 \times (n-2)$ board in $f_{n-2}$ ways.

**Total:** $f_{n-1} + f_{n-2}$

# Fibonacci identities

We have a new definition for Fibonacci:

$$f_n = \text{the number of square-domino tilings of a } 1 \times n \text{ board.}$$

This *combinatorial interpretation* of the Fibonacci numbers provides a framework to prove identities.

- Did you know that $f_{2n} = (f_n)^2 + (f_{n-1})^2$?

| $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ | $f_8$ | $f_9$ | $f_{10}$ | $f_{11}$ | $f_{12}$ | $f_{13}$ | $f_{14}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 5 | 8 | **13** | **21** | 34 | 55 | 89 | 144 | 233 | 377 | **610** |

$$f_{14} = f_7^2 + f_6^2$$

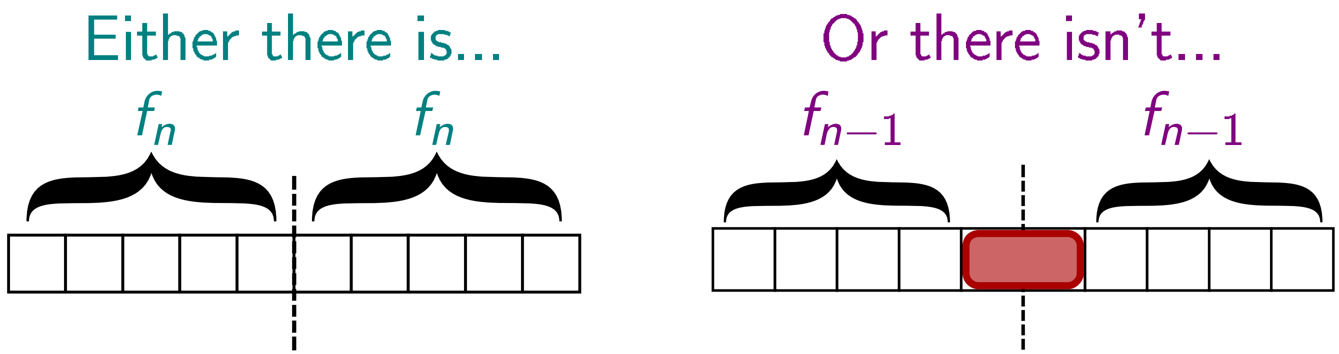
$$610 = 441 + 169$$

# Proof that $f_{2n} = (f_n)^2 + (f_{n-1})^2$

*Proof.* How many ways are there to tile a $1 \times (2n)$ board?

*Answer 1.* Duh, $f_{2n}$.

*Answer 2.* Ask whether there is a break in the middle of the tiling:




For a total of $(f_n)^2 + (f_{n-1})^2$ tilings.

We counted $f_{2n}$ in two different ways, so they must be equal. $\square$

**Further reading:**

Arthur T. Benjamin and Jennifer J. Quinn
Proofs that Really Count, MAA Press, 2003.