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- Finding the right set of objects is important (and difficult).

A Simple Combinatorial Proof

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Question: In how many ways can we choose from n club members a committee of k members with a chairperson?

Answer 1:

Answer 2:

Because the two quantities count the same set of objects in two different ways, the two answers are equal. $\square$

Pascal's Identity

Example. Prove *Theorem* 2.2.1: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Combinatorial Proof:

Question: In how many ways can we choose k flavors of ice cream if n different choices are available?

Answer 1:

Answer 2:

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Summing Binomial Coefficients

Example. Prove *Equation* (2.3): $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$.

Analytic Proof: ???

Combinatorial Proof:

Question: How many subsets of $\{1, 2, \ldots, n\}$ are there?

Answer 1: Condition on how many elements are in a subset.

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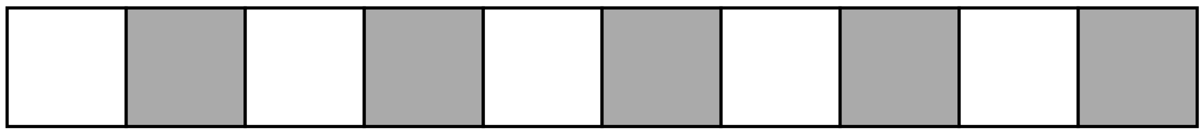
Tiling a board with dominos and squares

Question: How many ways are there to tile a $1 \times n$ board using only dominoes and squares?



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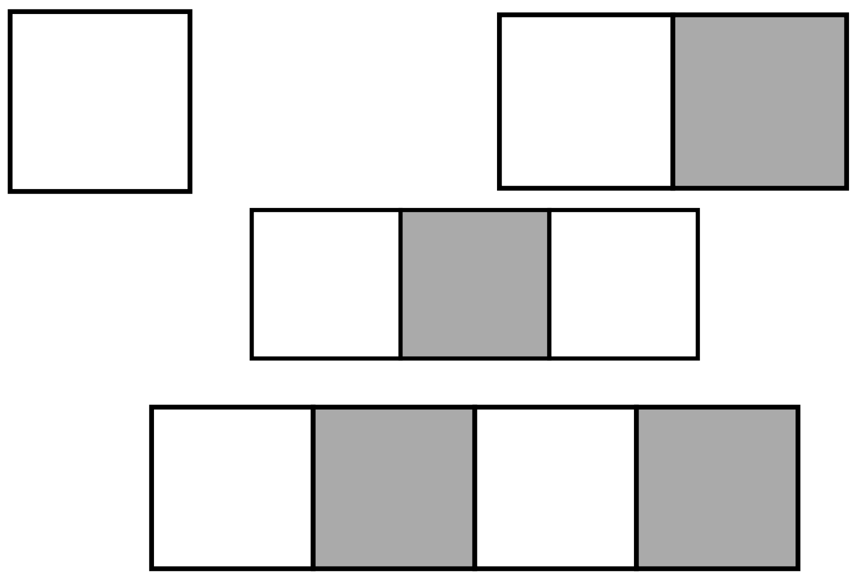

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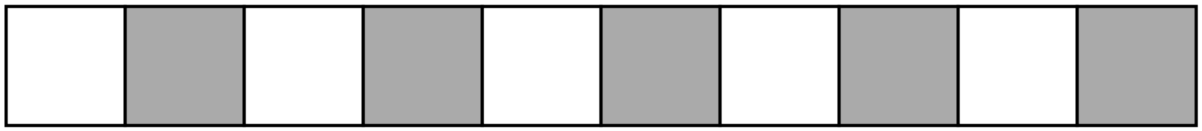

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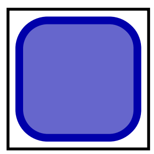

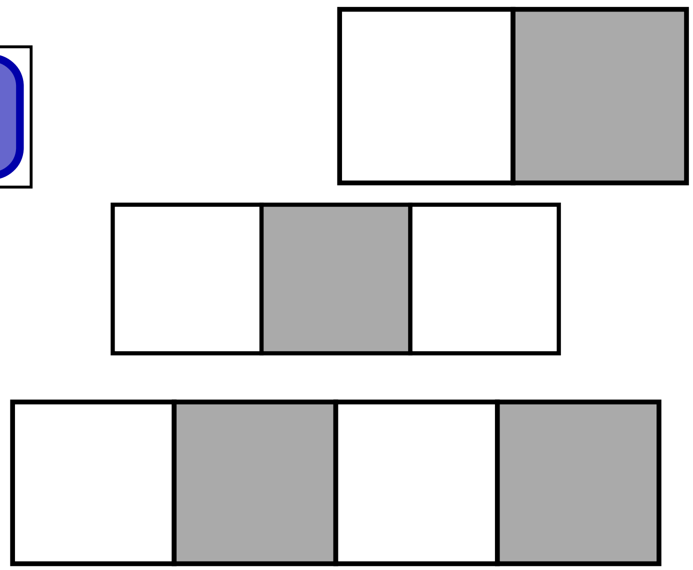

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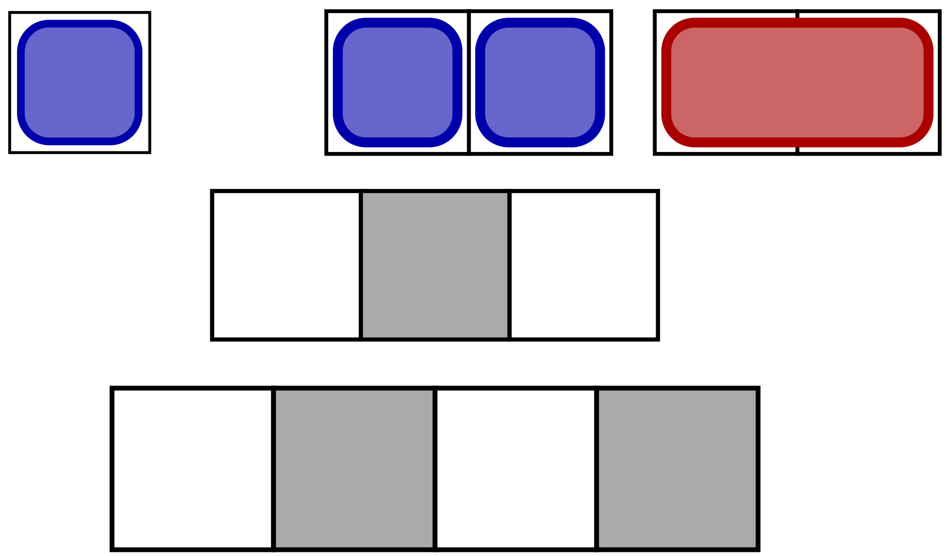

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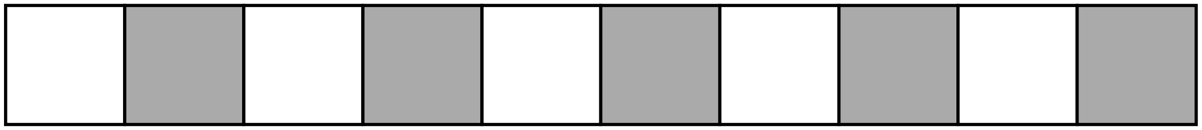

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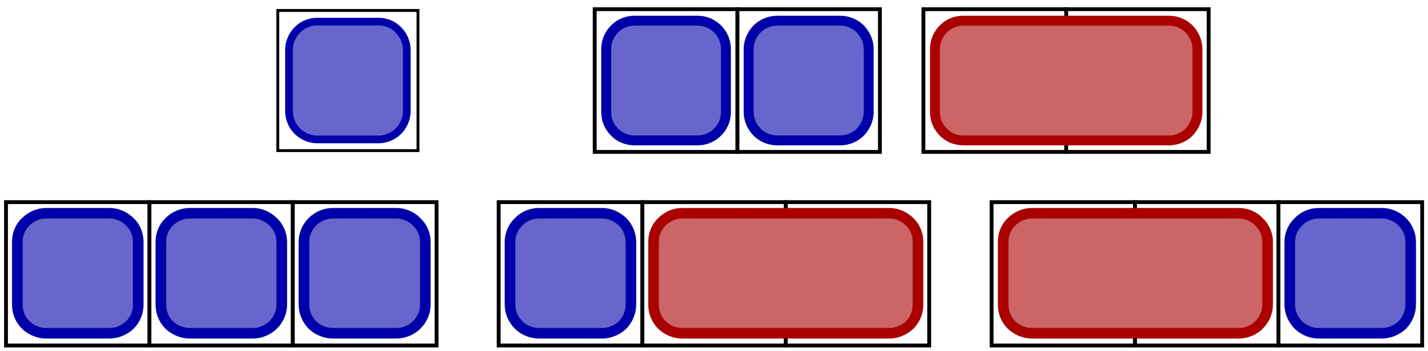

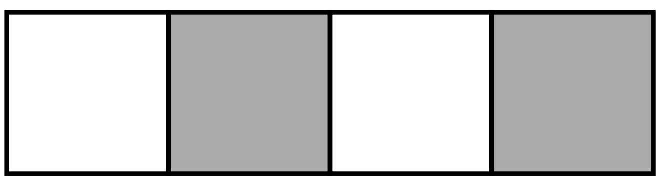

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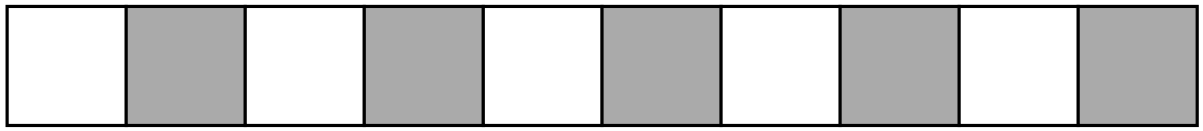

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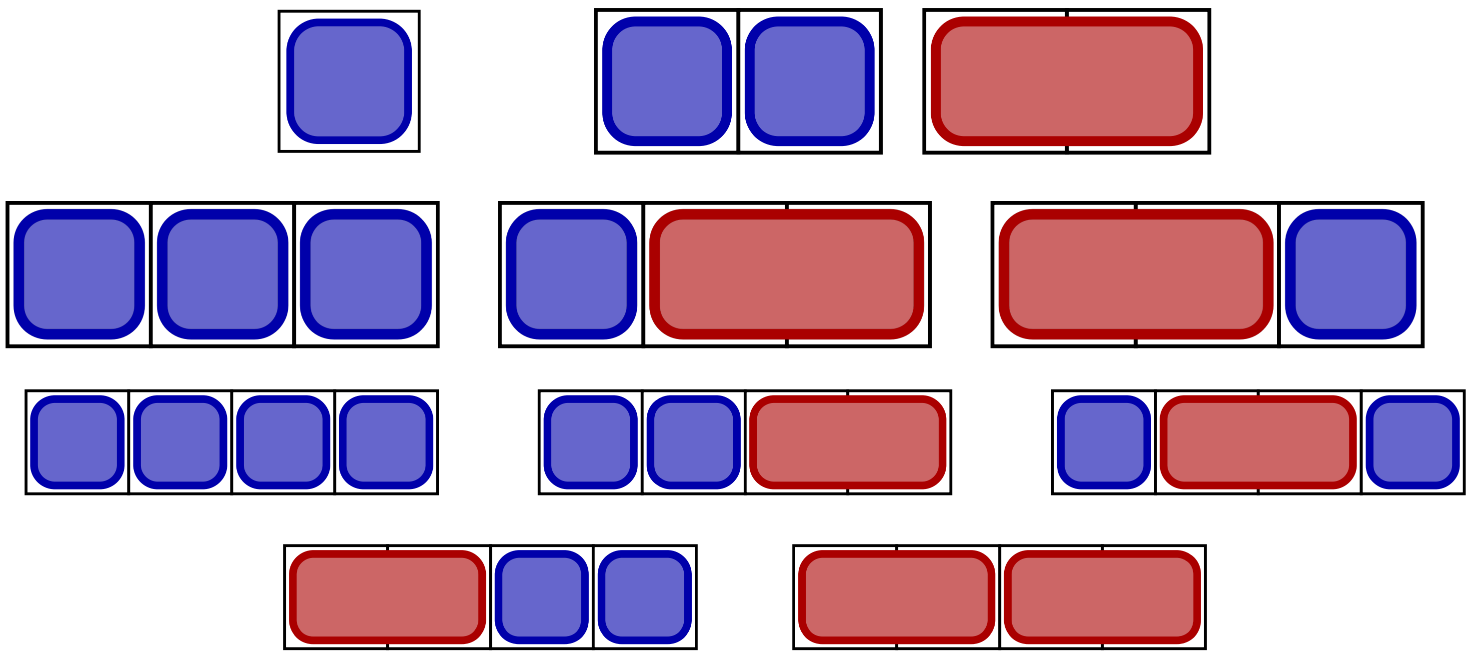

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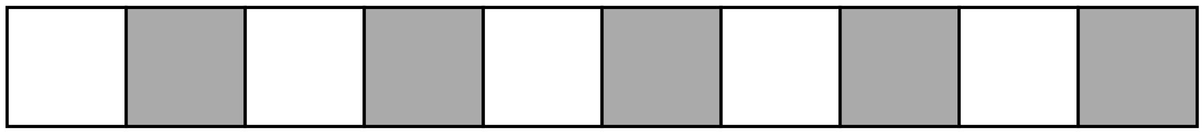

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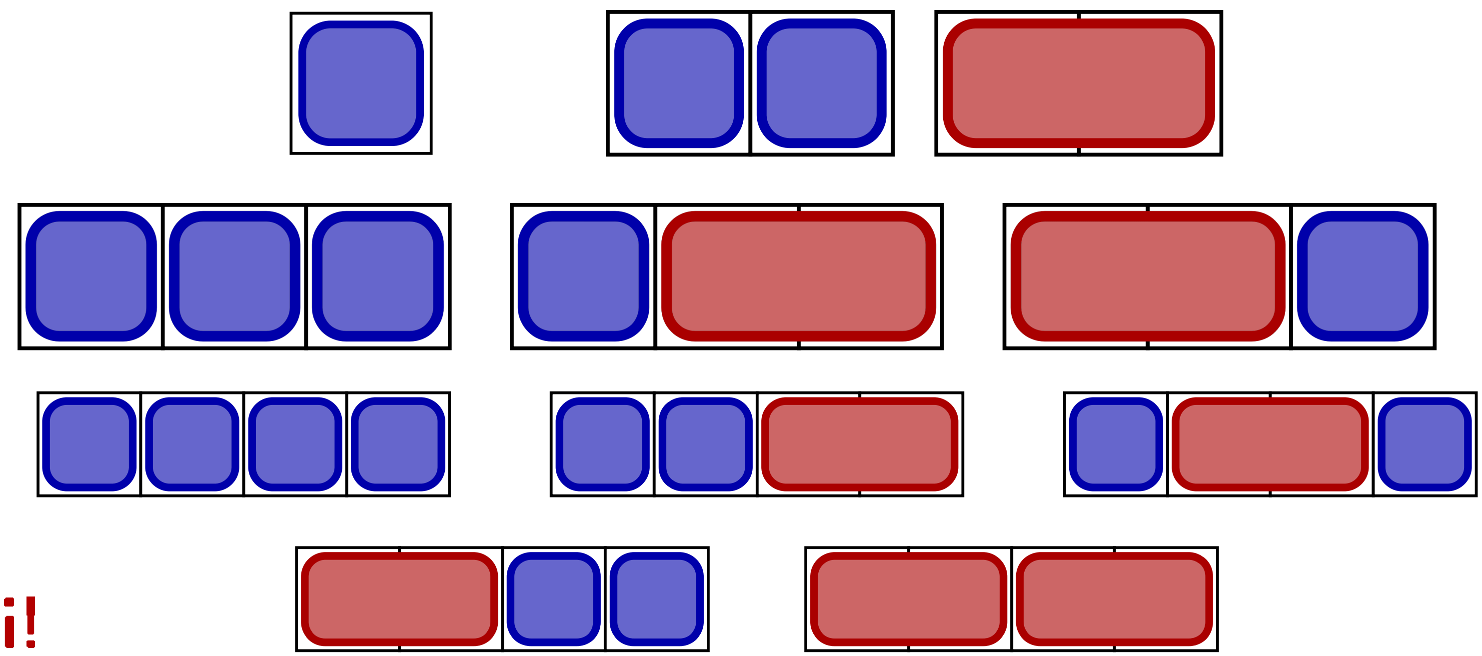

Fibonacci!

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Fibonacci numbers f_n satisfy

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Every tiling ends in either:

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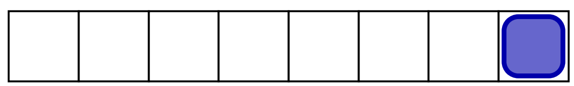

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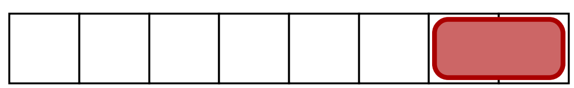

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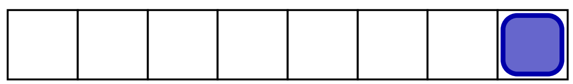
- a square 
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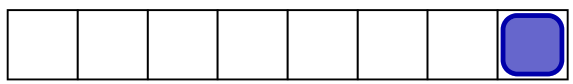
- a square 
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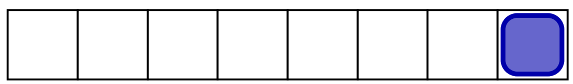
- a square 
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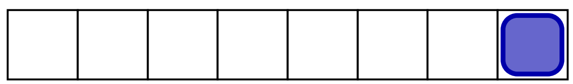
- a square 
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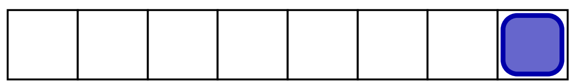
- a square 
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1	2	**3**	**5**	8	13	21	**34**	55	89	144	233	377	610

$$f_8 = f_4^2 + f_3^2$$

$$34 = 25 + 9$$

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$$f_{14} = f_7^2 + f_6^2$$

$$610 = 441 + 169$$

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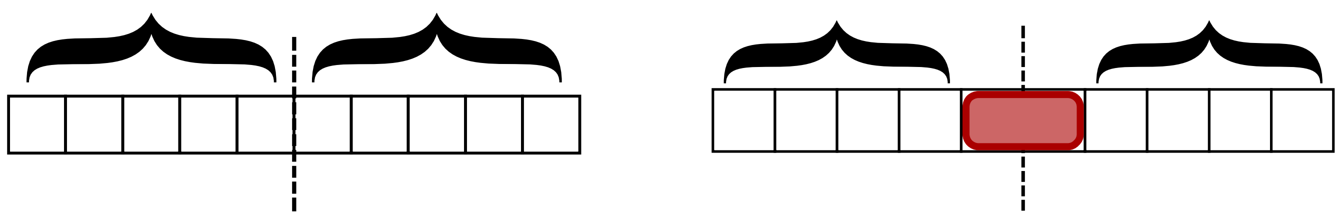
Either there is... Or there isn't...



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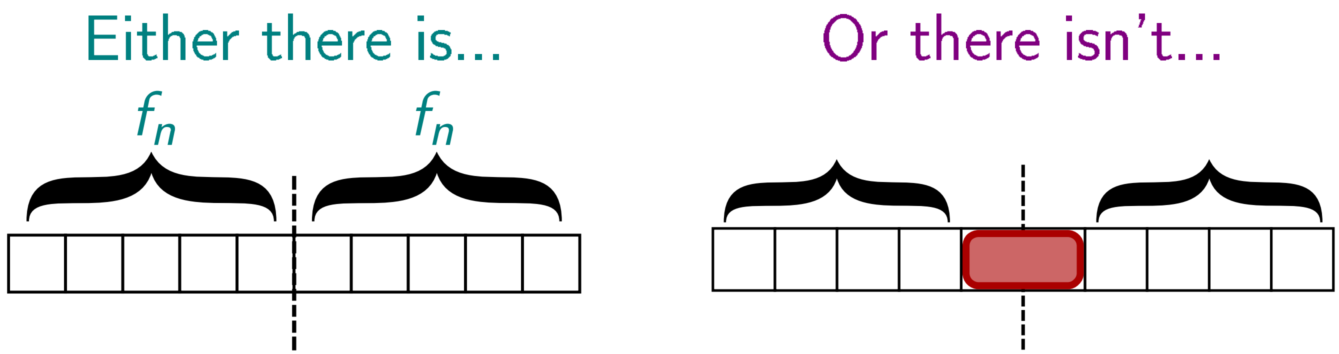

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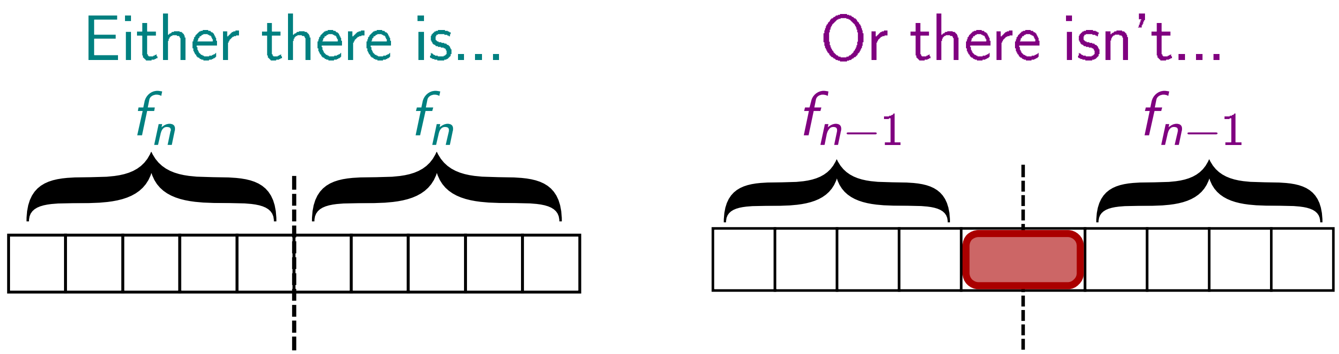

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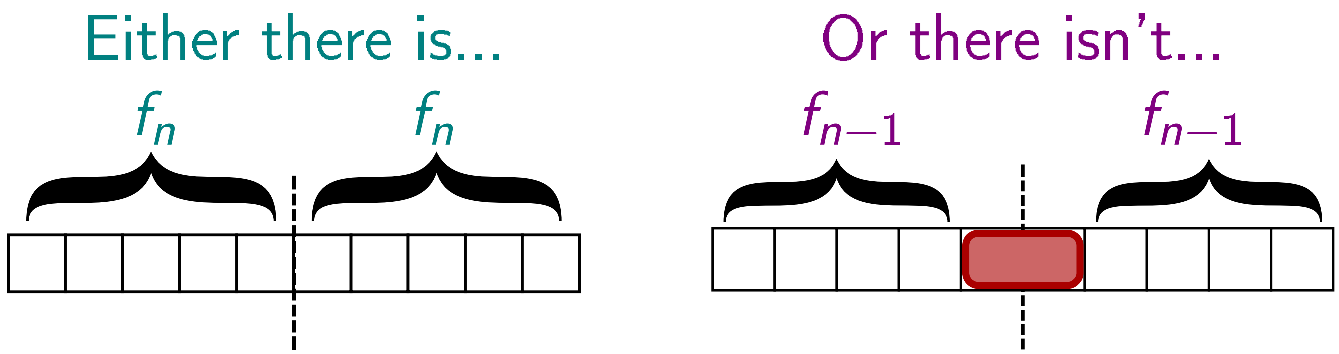

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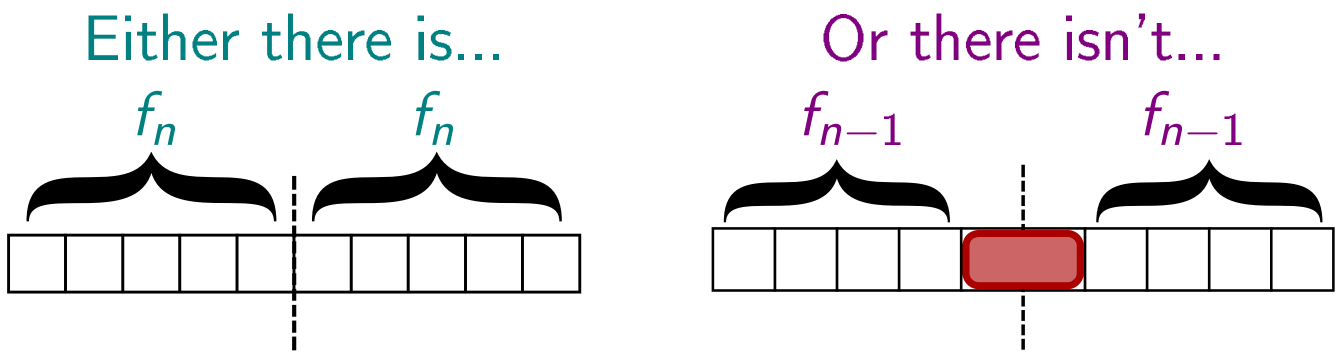

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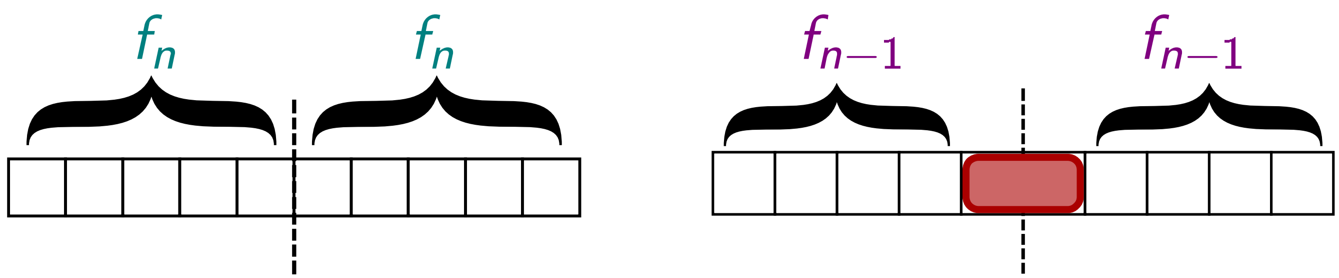

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Further reading:

Arthur T. Benjamin and Jennifer J. Quinn
Proofs that Really Count, MAA Press, 2003.