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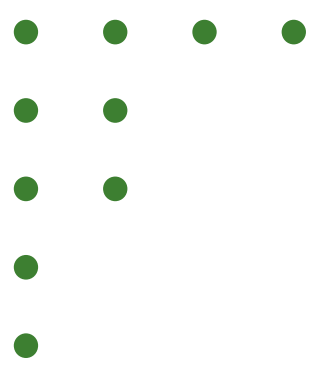
Example. The Ferrers diagram of $42211 \vdash 10$ is



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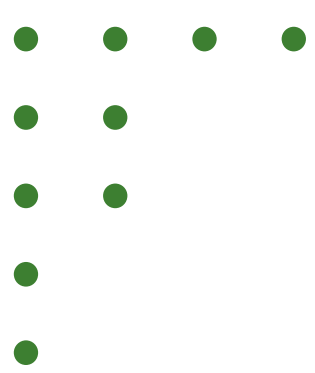

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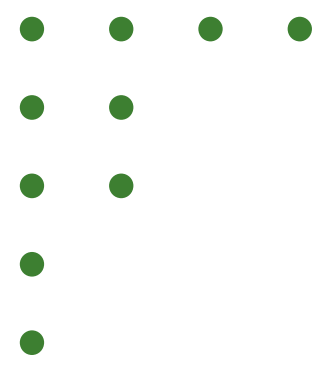

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Some partitions are **self-conjugate**, satisfying $\lambda = \lambda^c$.

A generating function for partitions

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- Understand each factor in the product well to find a generating function for a subset of partitions.
- The generating function is beautiful! But **no nice formula**!

A formula for integer partitions

Bruinier and Ono (2011) found an algebraic formula for the partition function $P(n)$ as a finite sum of algebraic numbers as follows. Define the weight-2 meromorphic modular form $F(z)$ by

$$F(z) = \frac{1}{2} \frac{E_2(z) - 2E_2(2z) - 3E_2(3z) + 6E_2(6z)}{\eta^2(z)\eta^2(2z)\eta^2(3z)\eta^3(6z)}, \tag{27}$$

were $q = e^{2\pi i z}$, $E_2(q)$ is an Eisenstein series, and $\eta(q)$ is a Dedekind eta function. Now define

$$R(z) = -\left(\frac{1}{2\pi i}\frac{d}{dz} + \frac{1}{2\pi y}\right) F(z), \tag{28}$$

where $z = x + iy$. Additionally let Q_n be any set of representatives of the equivalence classes of the integral binary quadratic form $Q(x, y) = ax^2 + bxy + cy^2$ such that $6 \mid a$ with $a > 0$ and $b \equiv 1 \pmod{12}$, and for each $Q(x, y)$, let α_Q be the so-called CM point in the upper half-plane, for which $Q(\alpha_Q, 1) = 0$. Then

$$P(n) = \frac{\mathrm{Tr}(n)}{24n - 1}, \tag{29}$$

where the trace is defined as

$$\mathrm{Tr}(n) = \sum_{Q \in Q_n} R(\alpha_Q). \tag{30}$$

Weisstein, Eric W. "Partition Function P."
From MathWorld—A Wolfram Web Resource.
`http://mathworld.wolfram.com/PartitionFunctionP.html`

Partitions: odd parts and distinct parts

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See, I told you they were equal. $\square$

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We use $P(n, *)$ to restrict partitions. Recall $P(n, k)$: exactly k parts.

Example. Prove a recurrence relation for $P(n, k)$:

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- If so, there are $P(n-1, k-1)$ partitions via the bijection

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The same bijection gives:

Theorem 4.4.2. _____ equals $P(n, \textit{largest part} \leq k)$.

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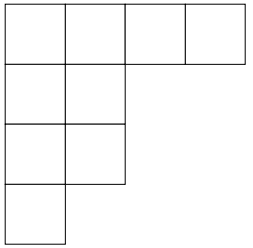

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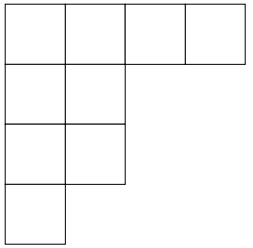

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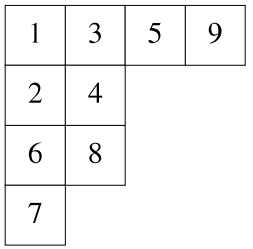

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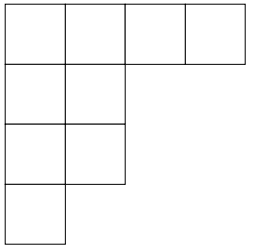

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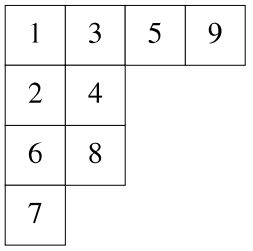

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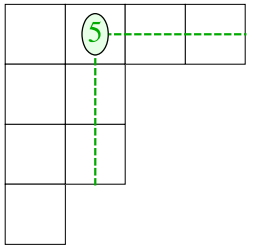

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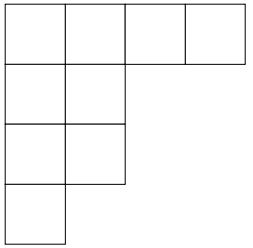

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2	4		
6	8		
7			

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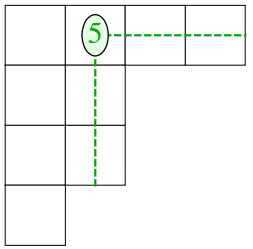

Question: How many SYT are there of shape $\lambda \vdash n$?

Answer: $$\frac{n!}{\prod_{(i,j)\in\lambda} h(i,j)}$$