Generating functions

"A generating function is a clothesline on which we hang up a sequence of numbers for display."

— *Generatingfunctionology*, H. S. Wilf

Definition: For any sequence $\{a_k\}_{k\geq 0} = a_0, a_1, a_2, a_3, \ldots$, its generating function is the formal power series

$$A(x) = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + \cdots = \sum_{k\geq 0} a_kx^k.$$

Example. Let f_k be the Fibonacci numbers starting $f_0 = f_1 = 1$. Then

$$F(x) = \sum_{k\geq 0} f_kx^k = 1 + 1x^1 + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + 21x^7 + \cdots.$$

This expression simplifies greatly. In fact,

$$F(x) = 1/\bigl(1 - x - x^2\bigr).$$

We will call this the compact form of the generating function.

Why Generating Functions?

We will use generating functions to:

- Find an exact formula for the terms of a sequence.
- Prove identities involving sequences.
- Understand partitions of integers.
- Use algebra to solve combinatorial problems.

Others use generating functions to:

- Use complex analysis to solve combinatorial problems.
- Understand the asymptotics of a sequence.
- Find averages and statistical properties.
- Understand **something** about a sequence.

Generating function example: Basketball

Example. In how many ways can a team score a total of six points in basketball? (Recall that a shot is worth either 1, 2, or 3 points.)

Solution. This is a partition of 6 into parts of size at most 3, so 7:

$$3+3 \qquad 3+2+1 \qquad 3+1+1+1 \qquad 2+2+2$$
$$2+2+1+1 \qquad 2+1+1+1+1 \qquad 1+1+1+1+1+1$$

What about 98 points?

Generating functions will help us keep track of the possibilities.

We'll first reanalyze the question of scoring six points and then generalize to larger numbers.

To start, break down the possible ways of getting six points total into one-point, two-point, and three-point shots.

Generating function example: Basketball

How many points could be scored using one-point shots?

0 pts	or	1 pt	or	2 pts	or	3 pts	or	4 pts	or	5 pts	or	6 pts
x^0	$+$	x^1	$+$	x^2	$+$	x^3	$+$	x^4	$+$	x^5	$+$	x^6

How many points could be scored using two-point shots?

How many points could be scored using three-point shots?

Multiply these algebraic expressions together:

$$1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 7x^6 + 7x^7 + 8x^8 + 8x^9 + 8x^{10} + 7x^{11} + 7x^{12} + 5x^{13} + 4x^{14} + 3x^{15} + 2x^{16} + x^{17} + x^{18}$$

and find the coefficient of the x^6 term.

Why does this work? A score a from 1-pt, b from 2-pt, c from 3-pt, gives a term in the product of $x^a x^b x^c = x^{a+b+c}$. Collecting like terms makes the coefficient of x^k the number of ways to score k points.

Generating function example: Basketball

In order to take into account all the ways to score 98 points, we include more terms in each factor:

One-point shots: $1 + x + x^2 + \cdots +$ $\qquad = \underline{\qquad\qquad}.$

Two-point shots: $1 + x^2 + x^4 + \cdots +$ $\qquad = \underline{\qquad\qquad}.$

Three-point shots: $1 + x^3 + x^6 + \cdots +$ $\qquad = \underline{\qquad\qquad}.$

Conclusion: The generating function for the number of ways to score any number of points in basketball is

$$b(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)}.$$

In order to use $b(x)$, we would need to extract coefficients of the Taylor expansion of $b(x)$ about $x = 0$.

- Do direct series manipulations.
- Use a computer! SAGE, Mathematica, Maple, Matlab, etc.

Notation: $[x^k]f(x)$ is the coefficient of x^k in the expansion of the generating function $f(x)$. Example. $[x^{98}]b(x) = 850.$

Example: Fruit baskets

Example. In how many ways we can create a fruit basket with n pieces of fruit, where we have an infinite supply of apples and bananas, with the added constraints:

- The number of apples is even.
- The number of bananas is a multiple of five.
- The number of oranges is at most four.
- The number of pears is zero or one.

Strategy. Write down a power series for each piece of fruit, multiply them together, and extract the coefficient of x^k.

Key series

⋆ Use these key series to collapse sums to compact forms or extract coefficients. ⋆

$$\frac{1}{1-x} = \sum_{k\geq 0} x^k \qquad \frac{1}{1-c\,x} = \sum_{k\geq 0} c^k x^k \qquad \frac{1}{1+x} = \sum_{k\geq 0} (-1)^k x^k$$

$$(1+x)^n = \sum_{k\geq 0} \binom{n}{k} x^k \qquad \frac{1}{(1-x)^n} = \sum_{k\geq 0} \left(\!\!\binom{n}{k}\!\!\right) x^k$$

$$\underbrace{(1+x)}_{1}\underbrace{(1+x)}_{2}\cdots\underbrace{(1+x)}_{n} \qquad \underbrace{(1+x+x^2+\cdots)}_{1}\cdots\underbrace{(1+x+x^2+\cdots)}_{n}$$

Manipulations on $A(x) = \sum_{k\geq 0} a_k x^k$

Example. Find the coefficient of x^9 in $\frac{1}{(1-4x)^{12}}$.

Question: Let $A(x) = x^b B(x)$. How can we simplify $[x^k]\big(A(x)\big)$?

Example. Calculate $[x^{10}]\left(\frac{x^4}{1+2x} + \frac{x^7}{1+x}\right)$.

Question: How can we simplify:

$$\sum_{k\geq 1} a_{k-1} x^k =$$

$$\sum_{k\geq 0} a_{k+1} x^k =$$

Example. Find the compact form of $\sum_{k\geq 2} (-3)^{k-2} x^k$.

Memories of calculus. . .

With formal power series, we interchange derivatives, integrals, sums.

$$\sum_{k\geq 0} \frac{x^{k+1}}{k+1} = \sum_{k\geq 0} \int_0^x x^k \, dx = \int_0^x \sum_{k\geq 0} x^k \, dx = \int_0^x \frac{1}{1-x} \, dx = -\ln|1-x|$$

$$\sum_{k\geq 0} kx^{k-1} = \sum_{k\geq 0} \frac{d}{dx} x^k = \frac{d}{dx} \sum_{k\geq 0} x^k = \frac{d}{dx} \frac{1}{1-x} = \frac{1}{(1-x)^2}$$

$$\sum_{k\geq 0} k^2 x^k =$$

If $A(x) = \sum_{k\geq 0} a_k x^k$, then $\sum_{k\geq 0} p(k) a_k x^k = p\left(x \frac{d}{dx}\right)(A(x))$

Multiplying two generating functions (Convolution)

Let $A(x) = \sum_{k \geq 0} a_k x^k$ and $B(x) = \sum_{k \geq 0} b_k x^k$.

Question: What is the coefficient of x^k in $A(x)B(x)$?

When expanding the product $A(x)B(x)$ we multiply terms $a_i x^i$ in A by terms $b_j x^j$ in B. This product contributes to the coefficient of x^k in AB only when ________________________.

Therefore, $A(x)B(x) = \sum_{k \geq 0} \Big(\underline{\qquad\qquad\qquad} \Big) x^k$

Example.
$[x^9] \frac{x^3(1+x)^4}{(1-2x)}$

Combinatorial interpretation of the convolution:

If a_k counts all "A" objects of "size" k, and
b_k counts all "B" objects of "size" k,

Then $[x^k]\big(A(x)B(x)\big)$ counts all pairs of objects (A, B) with *total* size k.

A Halloween Multiplication

Example. In how many ways can we fill a halloween bag w/30 candies, where for each of 20 BIG candy bars, we can choose at most one, and for each of 40 different small candies, we can choose as many as we like?

Big candy g.f.: $B(x) = (1+x)^{20} = \sum_{k=0}^{\infty} \binom{20}{k} x^k.$

Small candy g.f.: $S(x) = \frac{1}{(1-x)^{40}} = \sum_{k=0}^{\infty} \left(\!\binom{40}{k}\!\right) x^k.$

Total g.f.: $B(x)S(x) = \sum_{k=0}^{\infty} \left[\sum_{i=0}^{k} \binom{20}{i} \left(\!\binom{40}{k-i}\!\right) \right] x^k$

Conclusion: $[x^{30}]B(x)S(x) = \sum_{i=0}^{30} \binom{20}{i} \left(\!\binom{40}{30-i}\!\right)$

b_k counts (k big candies)

s_k counts (k small candies)

So, $[x^k]B(x)S(x)$ counts pairs of the form $\vee$ w/k total candies.

(some number of big candies, some number of small candies)

Example: Rolling dice

Example. When two standard six-sided dice are rolled, what is the distribution of the sums that appear?

Solution. The generating function for one die is $D(x) =$
Therefore, the distribution of sums for rolling two dice is

Question: What does $D(1)$ count?
Answer:

Example. Is it possible to relabel two six-sided dice differently to give the *exact same distribution* of sums?

Solution. Find two generating functions $F(x)$ and $G(x)$ such that $F(x)G(x) = D^2(x)$ and $F(1) = G(1) = 6$. Rearrange the factors:

$$\begin{aligned} D(x)^2 &= x^2(1+x)^2(1-x+x^2)^2(1+x+x^2)^2. \\ &= \left[x(1+x)(1+x+x^2)\right] \cdot \left[x(1-x+x^2)^2(1+x)(1+x+x^2)\right] \\ &= \left[x+2x^2+2x^3+x^4\right] \cdot \left[x+x^3+x^4+x^5+x^6+x^8\right] \end{aligned}$$

Die F: $\{1,2,2,3,3,4\}$ and die G: $\{1,3,4,5,6,8\}$

Vandermonde's Identity (p. 117)

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}$$

Combinatorial proof

Generating function proof