Generating functions

"A generating function is a clothesline on which we hang up a sequence of numbers for display."

— *Generatingfunctionology*, H. S. Wilf

Definition: For any sequence $\{a_k\}_{k\geq 0} = a_0, a_1, a_2, a_3, \ldots$, its generating function is the formal power series

$$A(x) = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + \cdots = \sum_{k\geq 0} a_kx^k.$$

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Example. Let f_k be the Fibonacci numbers starting $f_0 = f_1 = 1$. Then

$$F(x) = \sum_{k\geq 0} f_k x^k = 1 + 1x^1 + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + 21x^7 + \cdots .$$

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This expression simplifies greatly. In fact,

$$F(x) = 1/\left(1 - x - x^2\right).$$

We will call this the compact form of the generating function.

Why Generating Functions?

We will use generating functions to:

- Find an exact formula for the terms of a sequence.
- Prove identities involving sequences.
- Understand partitions of integers.
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Others use generating functions to:

- Use complex analysis to solve combinatorial problems.
- Understand the asymptotics of a sequence.
- Find averages and statistical properties.
- Understand **something** about a sequence.

Generating function example: Basketball

Example. In how many ways can a team score a total of six points in basketball? (Recall that a shot is worth either 1, 2, or 3 points.)

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Solution. This is a partition of 6 into parts of size at most 3, so 7:

$$3+3 \qquad 3+2+1 \qquad 3+1+1+1 \qquad 2+2+2$$
$$2+2+1+1 \qquad 2+1+1+1+1 \qquad 1+1+1+1+1+1$$

What about 98 points?

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Generating functions will help us keep track of the possibilities.

We'll first reanalyze the question of scoring six points and then generalize to larger numbers.

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What about 98 points?

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We'll first reanalyze the question of scoring six points and then generalize to larger numbers.

To start, break down the possible ways of getting six points total into one-point, two-point, and three-point shots.

Generating function example: Basketball

How many points could be scored using one-point shots?

0 pts	or	1 pt	or	2 pts	or	3 pts	or	4 pts	or	5 pts	or	6 pts
x^0	$+$	x^1	$+$	x^2	$+$	x^3	$+$	x^4	$+$	x^5	$+$	x^6

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How many points could be scored using two-point shots?

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$$x^0 + x^1 + x^2 + x^3 + x^4 + x^5 + x^6$$

How many points could be scored using two-point shots?

How many points could be scored using three-point shots?

Multiply these algebraic expressions together:

$$1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 7x^6 + 7x^7 + 8x^8 + 8x^9 + 8x^{10} + 7x^{11} + 7x^{12} + 5x^{13} + 4x^{14} + 3x^{15} + 2x^{16} + x^{17} + x^{18}$$

and find the coefficient of the x^6 term.

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Why does this work? A score a from 1-pt, b from 2-pt, c from 3-pt, gives a term in the product of $x^a x^b x^c = x^{a+b+c}$. Collecting like terms makes the coefficient of x^k the number of ways to score k points. $(x^{15}?)$

Generating function example: Basketball

In order to take into account all the ways to score 98 points, we include more terms in each factor:

One-point shots: $1 + x + x^2 + \cdots +$ $= \underline{\hspace{3cm}}.$

Two-point shots: $1 + x^2 + x^4 + \cdots +$ $= \underline{\hspace{3cm}}.$

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Conclusion: The generating function for the number of ways to score any number of points in basketball is

$$b(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)}.$$

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Notation: $[x^k]f(x)$ is the coefficient of x^k in the expansion of the generating function $f(x)$. Example. $[x^{98}]b(x) = 850$.

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- The number of apples is even.
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- The number of oranges is at most four.
- The number of pears is zero or one.

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- The number of apples is even.
- The number of bananas is a multiple of five.
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Strategy. Write down a power series for each piece of fruit, multiply them together, and extract the coefficient of x^k.

Key series

⋆ Use these key series to collapse sums to compact forms or extract coefficients. ⋆

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$$(1+x)^n = \sum_{k\geq 0} \binom{n}{k} x^k$$

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$$\underbrace{(1+x)}_{1}\underbrace{(1+x)}_{2}\cdots\underbrace{(1+x)}_{n} \qquad \underbrace{(1+x+x^2+\cdots)}_{1}\cdots\underbrace{(1+x+x^2+\cdots)}_{n}$$

Manipulations on $A(x) = \sum_{k \geq 0} a_k x^k$

Example. Find the coefficient of x^9 in $\frac{1}{(1-4x)^{12}}$.

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Example. Find the compact form of $\sum_{k \geq 2} (-3)^{k-2} x^k$.

Memories of calculus. . .

With formal power series, we interchange derivatives, integrals, sums.

$$\sum_{k\geq 0} \int_0^x x^k \, dx = \int_0^x \sum_{k\geq 0} x^k \, dx$$

$$\sum_{k\geq 0} \frac{d}{dx} x^k = \frac{d}{dx} \sum_{k\geq 0} x^k$$

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If $A(x) = \sum_{k\geq 0} a_k x^k$, then $\sum_{k\geq 0} p(k) a_k x^k = p\left(x \frac{d}{dx}\right)(A(x))$

Multiplying two generating functions (Convolution)

Let $A(x) = \sum_{k\geq 0} a_k x^k$ and $B(x) = \sum_{k\geq 0} b_k x^k$.

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Therefore, $A(x)B(x) = \sum_{k \geq 0} \Big(\underline{\hspace{4cm}} \Big) x^k$

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Combinatorial interpretation of the convolution:

If a_k counts all "A" objects of "size" k, and
b_k counts all "B" objects of "size" k,

Then $[x^k]\big(A(x)B(x)\big)$ counts all pairs of objects (A, B) with *total* size k.

A Halloween Multiplication

Example. In how many ways can we fill a halloween bag w/30 candies, where for each of 20 BIG candy bars, we can choose at most one, and for each of 40 different small candies, we can choose as many as we like?

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Total g.f.: $B(x)S(x) = \sum_{k=0}^{\infty} \left[\sum_{i=0}^{k} \binom{20}{i} \left(\!\binom{40}{k-i}\!\right) \right] x^k$

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Conclusion: $[x^{30}]B(x)S(x) = \sum_{i=0}^{30} \binom{20}{i} \left(\!\binom{40}{30-i}\!\right)$

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s_k counts (k small candies)

Total g.f.: $B(x)S(x) = \sum_{k=0}^{\infty} \left[\sum_{i=0}^{k} \binom{20}{i} \left(\!\binom{40}{k-i}\!\right) \right] x^k$

Conclusion: $[x^{30}]B(x)S(x) = \sum_{i=0}^{30} \binom{20}{i} \left(\!\binom{40}{30-i}\!\right)$

So, $[x^k]B(x)S(x)$ counts pairs of the form $\vee$ w/k total candies.

(some number of big candies, some number of small candies)

Example: Rolling dice

Example. When two standard six-sided dice are rolled, what is the distribution of the sums that appear?

Solution. The generating function for one die is $D(x) =$

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$$D(x)^2 = x^2(1+x)^2\left(1-x+x^2\right)^2\left(1+x+x^2\right)^2.$$

$$= \left[x(1+x)\left(1+x+x^2\right)\right] \cdot \left[x\left(1-x+x^2\right)^2(1+x)\left(1+x+x^2\right)\right]$$

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Die F: $\{1,2,2,3,3,4\}$ and die G: $\{1,3,4,5,6,8\}$

Vandermonde's Identity (p. 117)

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j} \binom{n}{k-j}$$

Combinatorial proof

Generating function proof