

## Powers of generating functions

A special case of convolution gives the coefficients of powers of a g.f.:

$$(A(x))^2 = \sum_{k \geq 0} \left( \sum_{i=0}^k a_i a_{k-i} \right) x^k = \sum_{k \geq 0} \left( \sum_{i_1+i_2=k} a_{i_1} a_{i_2} \right) x^k.$$

Similarly,

$$(A(x))^n = \sum_{k \geq 0} \left( \sum_{i_1+i_2+\dots+i_n=k} a_{i_1} a_{i_2} \cdots a_{i_n} \right) x^k.$$

### Conclusion:

$[x^k](A(x))^n$  counts *sequences* of objects  $(A_1, A_2, \dots, A_n)$ , all of type  $A$ , with a total size (summed over all objects) of  $k$ .

**Example.** What is the generating function for the number of points that a basketball team can score if they hit a sequence of 10 baskets?

In how many ways can they score 20 points in those 10 baskets?

# Compositions

*Question:* In how many ways can we write a positive integer  $n$  as a sum of positive integers?

If order **doesn't** matter:

A **partition**:  $n = p_1 + p_2 + \cdots + p_\ell$  for positive integers  $p_1, p_2, \dots, p_\ell$  satisfying  $p_1 \geq p_2 \geq \cdots \geq p_\ell$ .

If order **does** matter:

A **composition**:  $n = i_1 + i_2 + \cdots + i_\ell$  for positive integers  $i_1, i_2, \dots, i_\ell$  with no restrictions.

*Example.* There are  $2^{n-1}$  compositions of  $n$ . When  $n = 4$ :

4  
3 + 1  
2 + 2  
2 + 1 + 1  
1 + 1 + 1 + 1

# Compositions of Generating Functions

*Question:* Let  $F(x) = \sum_{n \geq 0} f_n x^n$  and  $G(x) = \sum_{n \geq 0} g_n x^n$ .

What can we learn about the composition  $H(x) = F(G(x))$ ?

Investigate  $F(x) = 1/(1-x)$ .

$$H(x) = F(G(x)) = \frac{1}{1 - G(x)} = \dots$$

- ▶ This is an infinite sum of (likely infinite) power series. **Is this OK?**
- ▶ The constant term  $h_0$  of  $H(x)$  only makes sense if \_\_\_\_\_ .
- ▶ This implies that  $x^n$  divides  $G(x)^n$ .

Hence, there are at most  $n-1$  summands which contain  $x^{n-1}$ .

We conclude that the infinite sum makes sense.

For a general composition with  $g_0 = 0$ ,

$$F(G(x)) = \sum_{n \geq 0} f_n G(x)^n = f_0 + f_1 G(x) + f_2 G(x)^2 + f_3 G(x)^3 + \dots$$

# Compositions. of. Generating Functions.

Interpreting  $\frac{1}{1 - G(x)} = 1 + G(x)^1 + G(x)^2 + G(x)^3 + \dots$ :

**Recall:** The generating function  $G(x)^n$  counts sequences of length  $n$  of objects  $(G_1, G_2, \dots, G_n)$ , each of type  $G$ , and the coefficient  $[x^k](G(x)^n)$  counts those  $n$ -sequences that have **total size** equal to  $k$ .

**Conclusion:** As long as  $g_0 = 0$ , then  $1 + G(x)^1 + G(x)^2 + G(x)^3 + \dots$  counts sequences of **any length** of objects of type  $G$ , and the coefficient  $[x^k]\frac{1}{1 - G(x)}$  counts those that have **total size** equal to  $k$ .

**Alternatively:** Interpret  $[x^k]\frac{1}{1 - G(x)}$  thinking of  $k$  as this **total size**. First, find **all ways** to break down  $k$  into integers  $i_1 + \dots + i_\ell = k$ . Then create **all sequences** of objects of type  $G$  in which object  $j$  has size  $i_j$ .

**Think:** A composition of generating functions equals a composition. of. generating. functions.

## An Example, Compositions

**Example.** How many compositions of  $k$  are there?

**Solution.** A composition of  $k$  corresponds to a sequence  $(i_1, \dots, i_\ell)$  of positive integers (of any length) that sums to  $k$ .

The objects in the sequence are positive integers; we need the g.f. that counts how many positive integers there are with “size  $i$ ”.

What does size correspond to?

How many have value  $i$ ? Exactly one: the number  $i$ .

So the generating function for our objects is

$$G(x) = 0 + 1x^1 + 1x^2 + 1x^3 + 1x^4 + \dots = \underline{\hspace{10em}}.$$

We conclude that the generating function for compositions is

$$H(x) = \frac{1}{1-G(x)} =$$

So the number of compositions of  $n$  is

## A Composition Example

**Example.** How many ways are there to take a line of  $k$  soldiers, divide the line into non-empty platoons, and from each platoon choose one soldier in that platoon to be a leader?

**Solution.** A soldier assignment corresponds to a sequence of platoons of size  $(i_1, \dots, i_\ell)$ .

Given  $i$  soldiers in a platoon, in how many ways can we assign the platoon a leader? \_\_\_\_\_

Therefore  $G(x) =$

And the generating function for such a military breakdown is

$$H(x) = \frac{1}{1 - G(x)} = \frac{1 - 2x + x^2}{1 - 3x + x^2}$$

