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Example. What is the generating function for the number of points that a basketball team can score if they hit a sequence of 10 baskets?

In how many ways can they score 20 points in those 10 baskets?

Compositions

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Example. There are 2^{n-1} compositions of n. When $n = 4$:

$$4$$
$$3 + 1$$
$$2 + 2$$
$$2 + 1 + 1$$
$$1 + 1 + 1 + 1$$

Compositions of Generating Functions

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For a general composition with $g_0 = 0$,

$$F(G(x)) = \sum_{n\geq 0} f_n G(x)^n = f_0 + f_1 G(x) + f_2 G(x)^2 + f_3 G(x)^3 + \cdots .$$

Compositions. of. Generating Functions.

Interpreting $\dfrac{1}{1 - G(x)} = 1 + G(x)^1 + G(x)^2 + G(x)^3 + \cdots$:

Recall: The generating function $G(x)^n$ counts sequences of length n of objects $(G_1, G_2, \ldots, G_n)$, each of type G, and the coefficient $[x^k]\big(G(x)^n\big)$ counts those n-sequences that have total size equal to k.

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Conclusion: As long as $g_0 = 0$, then $1 + G(x)^1 + G(x)^2 + G(x)^3 + \cdots$ counts sequences of **any length** of objects of type G, and the coefficient $[x^k]\frac{1}{1-G(x)}$ counts those that have total size equal to k.

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Think: A composition of generating functions equals
a composition. of. generating. functions.

An Example, Compositions

Example. How many compositions of k are there?

Solution. A composition of k corresponds to a sequence $(i_1, \ldots, i_\ell)$ of positive integers (of any length) that sums to k.

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So the generating function for our objects is
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And the generating function for such a military breakdown is

$$H(x) = \frac{1}{1 - G(x)} = \frac{1 - 2x + x^2}{1 - 3x + x^2}$$

Domino Tilings

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Another way to see this:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}
$G(x)^0 =$													
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