

# Catalan Numbers

$c_0$	$c_1$	$c_2$	$c_3$	$c_4$	$c_5$	$c_6$	$c_7$	$c_8$	$c_9$	$c_{10}$
1	1	2	5	14	42	132	429	1430	4862	16796

On-Line Encyclopedia of Integer Sequences, <http://oeis.org/>

$$c_n = \frac{1}{n+1} \binom{2n}{n}.$$

Richard Stanley has compiled a list of comb. interpretations of Catalan numbers. List numbered (a) to (z), ... (a<sup>8</sup>) to (y<sup>8</sup>).

Now a book!

triangulations  
of an  $(n+2)$ -gon

lattice paths from  $(0,0)$   
to  $(n,n)$  above  $y = x$

sequences with  $n+1$ 's,  $n-1$ 's  
with positive partial sums

multiplication schemes  
to multiply  $n+1$  numbers

# Catalan Number Interpretations

When  $n = 3$ , there are  $c_3 = 5$  members of these families of objects:

1. Triangulations of an  $(n + 2)$ -gon
2. Lattice paths from  $(0, 0)$  to  $(n, n)$  staying above  $y = x$
3. Sequences of length  $2n$  with  $n + 1$ 's and  $n - 1$ 's such that every partial sum is  $\geq 0$
4. Ways to multiply  $n + 1$  numbers together two at a time.

# Catalan Bijections

We claim that these objects are all counted by the Catalan numbers. So there should be **bijections** between the sets!

Bijection 1: 

triangulations of an $(n+2)$ -gon
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 $\longleftrightarrow$ 

multiplication schemes to multiply $n + 1$ numbers
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Rule: Label all but one side of the  $(n + 2)$ -gon in order. Work your way in from the outside to label the interior edges of the triangulation: When you know two sides of a triangle, the third edge is the product of the two others. Determine the mult. scheme on the last edge.

# Catalan Bijections

Bijection 2: 

multiplication schemes to multiply $n + 1$ #'s
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 $\longleftrightarrow$ 

seqs with $n + 1$ 's, $n - 1$ 's with positive partial sums
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Rule: Place dots to represent multiplications. Ignore everything except the dots and right parentheses. Replace the dots by  $+1$ 's and the parentheses by  $-1$ 's.

# Catalan Bijections

Bijection 3: 

seqs with $n$ $+1$ 's, $n$ $-1$ 's with positive partial sums
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 $\longleftrightarrow$ 

lattice paths $(0, 0)$ to $(n, n)$ above $y = x$
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A sequence of  $+$ 's and  $-$ 's converts to a sequence of  $N$ 's and  $E$ 's, which is a path in the lattice.

# Catalan Numbers

The underlying reason why so many combinatorial families are counted by the Catalan numbers comes back to the **generating function equation** that  $C(x)$  satisfies:

$$C(x) = 1 + xC(x)^2.$$

**Example.**

triangulations  
of an  $(n+2)$ -gon

Here,  $x$  represents  
one side of the polygon

Either the triangulation has a side or not.

1. No side: Empty triangulation (of *digon*):  $x^0$ .
2. Every other triangulation has one side ( $x$  contribution) and is a **sequence of** two other triangulations  $C(x)^2$ .

# Catalan Numbers

Example.

lattice paths  $(0, 0)$  to  $(n, n)$  above  $y = x$

Here,  $x$  represents an up-step down-step pair.

Either the lattice path starts with a vertical step or not.

1. No step: Empty lattice path:  $x^0$ .
2. Every other lattice path has one vertical step up from diag. and a first horizontal step returning to diag. ( $x$  contribution).  
 “Between the  $V$  &  $H$  steps” and “after the  $H$  step”  
 is a sequence of two lattice paths  $C(x)^2$ .

Therefore,  $C(x) = 1 + xC(x)^2$ .

## A formula for the Catalan Numbers

Solve the generating function equation to find  $C(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}$ .  
Do we take the positive or negative root? Check  $x = 0$ .

Now extract coefficients to prove the formula for  $c_n$ .

**Claim:**  $\sqrt{1 - 4x} = 1 + \sum_{k \geq 1} \frac{-2}{k} \binom{2(k-1)}{k-1} x^k$ . (Next slide.)

**Conclusion.**  $\frac{1}{2x}(1 - \sqrt{1 - 4x}) = -\frac{1}{2x} \sum_{k \geq 1} \frac{-2}{k} \binom{2(k-1)}{k-1} x^k$   
 $= \sum_{k \geq 1} \frac{1}{k} \binom{2(k-1)}{k-1} x^{k-1}$   
 $= \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} x^n$

Therefore,  $c_n = \frac{1}{n+1} \binom{2n}{n}$ .

# Expansion of $\sqrt{1-4x}$

What is the power series expansion of  $\sqrt{1-4x}$ ?

$$\sqrt{1-4x} = ((-4x) + 1)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} (-4x)^k \quad \text{Expand } \binom{1/2}{k}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-k+1)}{k!} (-4x)^k \quad \text{Denom. of } \frac{1}{2}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{\frac{1}{2}(-\frac{1}{2})\cdots(-\frac{2k-3}{2})}{k!} (-1)^k 4^k x^k \quad \text{Factor } -2\text{'s}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(1)\cdots(2k-3)}{k!2^k} (-1)^k 4^k x^k \quad \text{Simplify; rewrite prod.}$$

$$= 1 + \sum_{k=1}^{\infty} -\frac{1\cdot 2\cdot 3\cdot 4\cdots(2k-3)\cdot(2k-2)}{k!\cdot 2\cdot 4\cdots(2k-2)} 2^k x^k \quad \text{Write as factorials}$$

$$= 1 + \sum_{k=1}^{\infty} -\frac{(2k-2)!}{k!(2^{k-1})1\cdot 2\cdots(k-1)} 2^k x^k$$

$$= 1 + \sum_{k=1}^{\infty} \frac{-2}{k} \frac{(2k-2)!}{(k-1)!(k-1)!} x^k$$

$$= 1 + \sum_{k=1}^{\infty} \frac{-2}{k} \binom{2(k-1)}{k-1} x^k$$