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Therefore, $A(x)B(x) = \sum_{k \geq 0} \Bigg(\underline{\qquad\qquad\qquad} \Bigg) x^k$

Example.

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Combinatorial interpretation of the convolution:

If a_k counts all "A" objects of "size" k, and
b_k counts all "B" objects of "size" k,

Then $[x^k]\big(A(x)B(x)\big)$ counts all pairs of objects (A, B) with *total* size k.

A Halloween Multiplication

Example. In how many ways can we fill a halloween bag w/30 candies, where for each of 20 BIG candy bars, we can choose at most one, and for each of 40 different small candies, we can choose as many as we like?

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So, $[x^k]B(x)S(x)$ counts pairs of the form $\vee$ w/k total candies.
(some number of big candies, some number of small candies)

Example: Rolling dice

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Die F: $\{1,2,2,3,3,4\}$ and die G: $\{1,3,4,5,6,8\}$

Vandermonde's Identity (p. 117)

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j} \binom{n}{k-j}$$

Combinatorial proof

Generating function proof

Powers of generating functions

A special case of convolution gives the coefficients of powers of a g.f.:

$$\big(A(x)\big)^2 = \sum_{k \geq 0} \left(\sum_{i=0}^{k} a_i a_{k-i} \right) x^k$$

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Example. What is the generating function for the number of points that a basketball team can score if they hit a sequence of 10 baskets?

In how many ways can they score 20 points in those 10 baskets?

Compositions

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If order **doesn't** matter:

A **partition**: $n = p_1 + p_2 + \cdots + p_\ell$ for positive integers $p_1, p_2 \ldots, p_\ell$ satisfying $p_1 \geq p_2 \geq \cdots \geq p_\ell$.

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Example. There are 2^{n-1} compositions of n. When $n = 4$:

$$4$$
$$3+1$$
$$2+2$$
$$2+1+1$$
$$1+1+1+1$$

Compositions of Generating Functions

Question: Let $F(x) = \sum_{n\geq 0} f_n x^n$ and $G(x) = \sum_{n\geq 0} g_n x^n$.
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For a general composition with $g_0 = 0$,

$$F(G(x)) = \sum_{n\geq 0} f_n G(x)^n = f_0 + f_1 G(x) + f_2 G(x)^2 + f_3 G(x)^3 + \cdots .$$

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Interpreting $\dfrac{1}{1 - G(x)} = 1 + G(x)^1 + G(x)^2 + G(x)^3 + \cdots$:

Recall: The generating function $G(x)^n$ counts sequences of length n of objects $(G_1, G_2, \ldots, G_n)$, each of type G, and the coefficient $[x^k]\big(G(x)^n\big)$ counts those n-sequences that have total size equal to k.

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Conclusion: As long as $g_0 = 0$, then $1 + G(x)^1 + G(x)^2 + G(x)^3 + \cdots$ counts sequences of **any length** of objects of type G, and the coefficient $[x^k]\frac{1}{1-G(x)}$ counts those that have total size equal to k.

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Alternatively: Interpret $[x^k]\frac{1}{1-G(x)}$ thinking of k as this *total size*. First, find **all ways** to break down k into integers $i_1 + \cdots + i_\ell = k$. Then create **all sequences** of objects of type G in which object j has size i_j.

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First, find **all ways** to break down k into integers $i_1 + \cdots + i_\ell = k$.
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Think: A composition of generating functions equals
a composition. of. generating. functions.

An Example, Compositions

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Solution. A composition of k corresponds to a sequence $(i_1, \ldots, i_\ell)$ of positive integers (of any length) that sums to k.

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So the number of compositions of n is

A Composition Example

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Given i soldiers in a platoon, in how many ways can we assign the platoon a leader? __________

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Given i soldiers in a platoon, in how many ways can we assign the platoon a leader? __________

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A Composition Example

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And the generating function for such a military breakdown is

$$H(x) = \frac{1}{1 - G(x)} = \frac{1 - 2x + x^2}{1 - 3x + x^2}$$

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Another way to see this:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}
$G(x)^0 =$													
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$1/(1 - G(x)) =$													