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The cardinality of a set is a combinatorial statistic on $\mathcal{S}$.

$$\begin{array}{cccc} \big|\emptyset\big| = 0 & \big|\{1\}\big| = 1 & \big|\{2\}\big| = 1 & \big|\{3\}\big| = 1 \\ \big|\{1,2\}\big| = 2 & \big|\{1,3\}\big| = 2 & \big|\{2,3\}\big| = 2 & \big|\{1,2,3\}\big| = 3 \end{array}$$

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Combinatorial statistics provide a *refinement* of counting.

less information ⟵ ⟶ *more information*

counting — **statistics** — **complete enumeration**

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counting

8

statistics

0	1	2	3
1	3	3	1

complete enumeration

$\emptyset$ $\{1\}$ $\{2\}$ $\{3\}$

$\{1,2\}$ $\{1,3\}$ $\{2,3\}$ $\{1,2,3\}$

Statistics and Permutations

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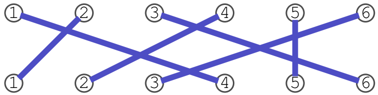
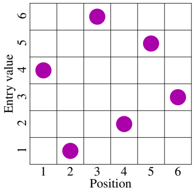
String diagram:



(only two crossings at a time)

Matrix-like diagram:



Descent statistic

Definition: Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation.

A **descent** is a position i such that $\pi_i > \pi_{i+1}$.

Define $\text{des}(\pi)$ to be the **number of descents** in π.

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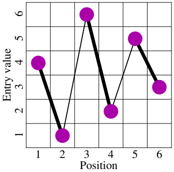

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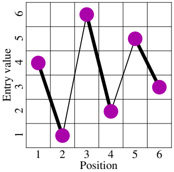

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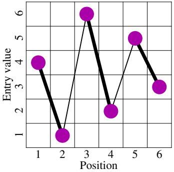

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$n \backslash d$	0	1	2	3	4
1	1				
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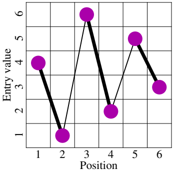

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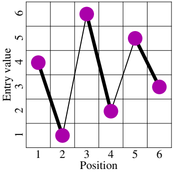

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These are the **Eulerian numbers**.

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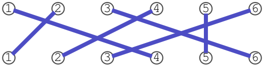

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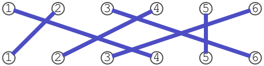

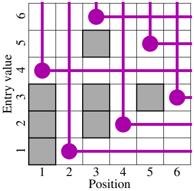

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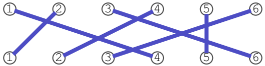

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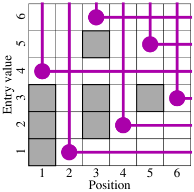

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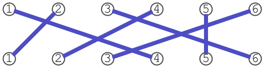

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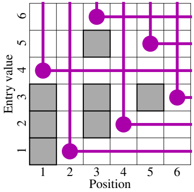

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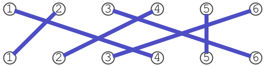

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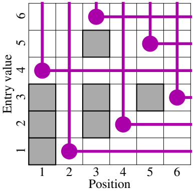

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The inversion number is a good way to count how "far away" a permutation is from the identity.

Major index

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Define maj(π), the **major index** of π, to be sum of the descents of π.

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A statistic that has the same distribution as inv is called **Mahonian**.

q-analogs

Definition: A q-**analog** of a number c is an expression $f(q)$ such that $\lim_{q\to 1} f(q) = c$.

Example. $\dfrac{1-q^n}{1-q} = \left(1 + q + q^2 + \cdots + q^{n-2} + q^{n-1}\right)$ is a q-analog of n because $\lim_{q\to 1} \frac{1-q^n}{1-q} = n$.

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Conjecture: $\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} =$

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Inversion statistics

Question: What is the generating function $\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)}$?

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Claim: This equation makes sense when $q = 1$.

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Theorem: $\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} = [n]_q!$

Proof. There exists a bijection

$$\left\{ \begin{array}{c} \text{permutations} \\ \pi \in S_n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{lists } (a_1, \ldots, a_n) \\ \text{where } 0 \leq a_i \leq n - i \end{array} \right\}.$$

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$$\begin{aligned}\sum_{\pi\in S_n} q^{\mathrm{inv}(\pi)} &= \sum_{a_1=0}^{n-1}\sum_{a_2=0}^{n-2}\cdots\sum_{a_n=0}^{0} q^{a_1+a_2+\cdots+a_n}\\ &= \Big(\sum_{a_1=0}^{n-1} q^{a_1}\Big)\Big(\sum_{a_2=0}^{n-2} q^{a_2}\Big)\cdots\Big(\sum_{a_n=0}^{0} q^{a_n}\Big)\\ &= \quad [n]_q \qquad [n-1]_q \quad \cdots \qquad [1]_q \quad = [n]_q!\end{aligned}$$

Notes

We said that inv and maj are equidistributed. Two possible proofs:

- Find a bijection $f : S_n \to S_n$ such that $\mathrm{maj}(\pi) = \mathrm{inv}(f(\pi))$.
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Define $\text{inv}(\pi) = |\{i < j \colon \pi(i) > \pi(j)\}|$.

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This can also be used to give a q-analog of the Catalan numbers.

There's always more to learn!!!

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