Applications of abacus diagrams: Simultaneous core partitions, alcoves, and a major statistic

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Partitions

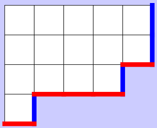
The **Young diagram** of $\lambda = (\lambda_1, \ldots, \lambda_k)$ has λ_i boxes in row i.

(James, Kerber) Create an **abacus diagram** from the boundary of λ.

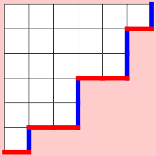
Abacus: Function $a : \mathbb{Z} \to \{\bullet, \text{␣}\}$. (Equivalence class...)

Partitions correspond to abacus diagrams.

(−9) (−8) (−7) (−6) (−5) (−4) (−3) −2 (−1) 0 1 2 (3) 4 (5) (6) 7 8 9



Partition



Self-conjugate partition

Self-conjugate partitions correspond to anti-symmetric abaci.

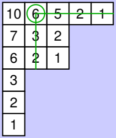
(−8) (−7) (−6) −5 (−4) −3 −2 (−1) (0) | 1 2 (3) (4) 5 (6) 7 8 9

Core partitions

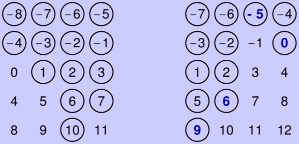
The **hook length** of a box = # boxes below + # boxes to right + box

λ is a ***t*-core** if no boxes have hook length $t \longleftrightarrow$ ***t*-flush** abacus

t-core partition

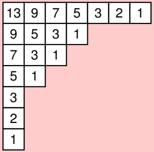


t-flush abacus (in runners)

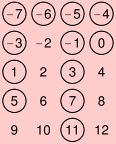


Normalized Balanced

Self-conj. t-core partition



t-flush antisymmetric abacus



Antisymmetry about $t/t+1$.

Simultaneity

Of interest: Partitions that are **both** s-core **and** t-core. $(s, t) = 1$

- Abaci that are both s-flush and t-flush.

There are infinitely many (self-conjugate) t-core partitions.

(s, t)-core partitions

9	6	5	3	2	1
5	2	1			
2					
1					

(Anderson, 2002):
\# (s, t)-core partitions
$\frac{1}{s+t}\binom{s+t}{s}$

Self-conj. (s, t)-core partitions

9	6	4	2	1
6	3	1		
4	1			
2				
1				

(Ford, Mai, Sze, 2009):
\# self-conj. (s, t)-core partitions
$\binom{s'+t'}{s'}$
where $s' = \left\lfloor \frac{s}{2} \right\rfloor$ and $t' = \left\lfloor \frac{t}{2} \right\rfloor$

Core partitions in the literature

Representation Theory: (origin)
t-cores label t-blocks of irreducible modular representations for S_n.
Nakayama cnj. Brauer-Robinson '47
s-c t-cores arise in rep. thy. of A_n.
- Readable survey by Kleshchev '10.

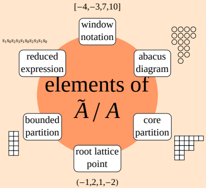
Numerical properties:
$c_t(n) = \#$ of t-core partitions of n.

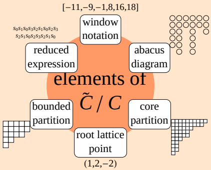
$$\sum_{n\geq 0} c_t(n) q^n = \prod_{n \geq 1} \frac{(1-q^{nt})^t}{1-q^n}$$

(↑ Olsson '76) (↓ Granville-Ono '96)
Positivity. $c_t(n) > 0$ $(t \geq 4)$.
Monotonicity? $c_{t+1}(n) \geq c_t(n)$

Modular forms:
G.f. for t-cores related to Dedekind's η-function, a mod. form of wt. $1/2$.

Coxeter groups: (↓ Lascoux '01)
$t+1$-cores $\longleftrightarrow$ coset reps in $\widetilde{A}_t/A_t$
- Keys: Bruhat order, Group action!



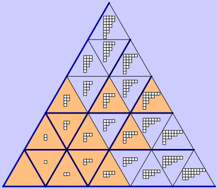


s-c t-cores $\longleftrightarrow$ coset reps in $\widetilde{C}_t/C_t$
One interpretation: Alcove geometry

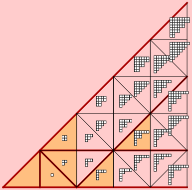
Alcove Geometry

Type A_t: generators $\{s_1, \ldots, s_t\}$
Group of permutations of $\{1, \ldots, t+1\}$.
Symmetries of regular simplex, dim. t.
Add one affine reflection s_0 to tile $\mathbb{R}^t$.
Dom. alcoves correspond to $t+1$-cores.
Overlay the m-Shi arrangement.
Which are representative alcoves?

Type A_2 alcoves



Type C_2 alcoves



Type C_t: generators $\{s_1, \ldots, s_t\}$
Group of signed permutations of $\{1, \ldots, t\}$.
Symmetries of cube or octa', dim. t.
Add one affine reflection s_0 to tile $\mathbb{R}^t$.
Dom. alcoves correspond to s.c. $2t$-cores.
Overlay the m-Shi arrangement.
Which are representative alcoves?

Alcoves and simultaneous cores

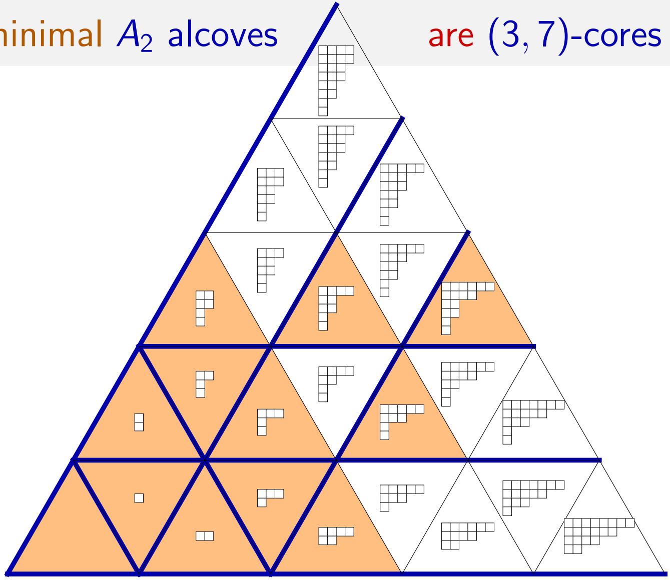
- For all dominant regions in m-Shi arrangement, the closest alcove to the origin is called m-**minimal**.
- For all *bounded* dominant regions in m-Shi arrangement, the furthest alcove from the origin is called m-**bounded**.

Theorem. (Fishel, Vazirani, '09–'10)
A_t alcove is m-minimal $\longleftrightarrow$ corresp. partition is $(t, tm+1)$-core.
A_t alcove is m-bounded $\longleftrightarrow$ corresp. partition is $(t, tm-1)$-core.

Theorem. (Armstrong, Hanusa, Jones, '13)
C_t alcove is m-minimal $\longleftrightarrow$ self-conjugate $(2t, 2tm+1)$-core.
C_t alcove is m-bounded $\longleftrightarrow$ self-conjugate $(2t, 2tm-1)$-core.

★ **Representative alcoves** correspond to **simultaneous cores**. ★

The 2-minimal A_2 alcoves are $(3,7)$-cores



Abaci to the rescue!

Proof sketch:

- m-minimal means that when it is reflected closer to to the origin, it must pass a hyperplane in the m-Shi arrangement.
- The equivalent abacus interpretation is that defining bead b_{i+1} is no more than m levels lower than b_i.
- Type A: So this t-flush abacus is also $(tm+1)$-flush.
  Type C: So this anti-symm. $2t$-flush abacus is also $(2tm+1)$-flush.
- A_t alcove is m-minimal $\longleftrightarrow$ $(t, tm+1)$-core.
  C_t alcove is m-minimal $\longleftrightarrow$ self-conj. $(2t, 2tm+1)$-core.

Numerical corollary: Agrees with (Athanasiadis, 2004).

- dominant A_t regions $\longleftrightarrow$ $(t, tm+1)$-cores. $\frac{1}{t+tm+1}\binom{t+tm+1}{t}$
  dominant C_t regions $\longleftrightarrow$ s-c. $(2t, 2tm+1)$-cores. $\binom{t+tm}{t}$

Abaci to the rescue!

Proof sketch:

- m-bounded means that when it is reflected further from the origin, it must pass a hyperplane in the m-Shi arrangement.
- The equivalent abacus interpretation is that defining bead b_{i+1} is no more than m levels higher than b_i.
- Type A: So this t-flush abacus is also $(tm-1)$-flush.
  Type C: So this anti-symm. $2t$-flush abacus is also $(2tm-1)$-flush.
- A_t alcove is m-bounded $\longleftrightarrow$ $(t, tm-1)$-core.
  C_t alcove is m-bounded $\longleftrightarrow$ s-c. $(2t, 2tm-1)$-core.

Numerical corollary: Agrees with (Athanasiadis, 2004).

- dom. bdd. A_t regions $\longleftrightarrow$ $(t, t-1)$-cores. $\frac{1}{t+tm-1}\binom{t+tm-1}{t}$
  dom. bdd. C_t regions $\longleftrightarrow$ s-c. $(2t, 2tm-1)$-cores. $\binom{t+tm-1}{t}$

Catalan numbers

Specializing the results of Anderson and Ford, Mai, and Sze,

# $(t, t+1)$-cores	# self-conj. $(2t, 2t+1)$-cores
$\frac{1}{2t+1}\binom{2t+1}{t} = \frac{1}{t+1}\binom{2t}{t}$	$\binom{2t}{t}$
A Catalan number! (of type A)	A Catalan number of type C

Question: Is there a simple statistic on simultaneous core partitions that gives us a q-analog of the Catalan numbers?

$$\sum_{\substack{\lambda \text{ is a} \\ (t,t+1)\text{-core}}} q^{\text{stat}(\lambda)} = \frac{1}{[t+1]_q}\begin{bmatrix} 2t \\ t \end{bmatrix}_q$$

$$\sum_{\substack{\lambda \text{ is a self-conj.} \\ (2t,2t+1)\text{-core}}} q^{\text{stat}(\lambda)} = \begin{bmatrix} 2t \\ t \end{bmatrix}_{q^2}$$

Answer: Yes. We will create an analog of the **major statistic**.

The major statistic

Given a permutation π of $\{1, \ldots, n\}$ written in one-line notation as $\pi = \pi_1\pi_2\cdots\pi_n$, the **major statistic** $\mathrm{maj}(\pi)$ is defined as the sum of the positions of the descents of π, in other words,

$$\mathrm{maj}(\pi) = \sum_{i:\pi_{i-1}>\pi_i} i.$$

Named in honor of Major Percy MacMahon who showed it has the same distribution as the statistic of the number of inversions:

$$\sum_{\pi\in S_n} q^{\mathrm{maj}(\pi)} = \sum_{\pi\in S_n} q^{\mathrm{inv}(\pi)}$$

A major statistic for simultaneous cores

Let λ be a $(t, t+1)$-core.
Define $b = (b_0, \dots, b_{t-1})$ where $b_i = \#\ 1^{\text{st}}$ col. boxes with hook length $\equiv i \bmod t$.
Define

$$\text{maj}(\lambda) = \sum_{i\,:\,b_{i-1} \geq b_i} (2i - b_i).$$

Theorem. (AHJ '13)

$$\sum_{\substack{\lambda \text{ is a} \\ (t,t+1)\text{-core}}} q^{\text{maj}(\lambda)} = \frac{1}{[t+1]_q} \begin{bmatrix} 2t \\ t \end{bmatrix}_q$$

Note: maj defined as a sum over descents in a sequence.

Let λ be a s-c. $(2t, 2t+1)$-core.
Define $b = (b_0, \dots, b_t)$ where $b_0 = 0$ and $b_i =$ $(\#$ diag. arms $\equiv i \bmod 2t) -$ $(\#$ diag. arms $\equiv 2t - i + 1 \bmod 2t)$
Define

$$\text{maj}(\lambda) = 2 \sum_{i\,:\,b_{i-1} \geq b_i} (2i - b_i - 1).$$

Theorem. (AHJ '13)

$$\sum_{\substack{\lambda \text{ is a self-conj.} \\ (2t,2t+1)\text{-core}}} q^{\text{maj}(\lambda)} = \begin{bmatrix} 2t \\ t \end{bmatrix}_{q^2}$$

A major statistic for abacus diagrams

Let λ be a $(t, t+1)$-core.

Read off the levels of the defining beads of the (normalized) abacus to give $b = (b_0, \ldots, b_{t-1})$.

Define

$$\text{maj}(\lambda) = \sum_{i\,:\,b_{i-1} \geq b_i} (2i - b_i).$$

Then

$$\sum_{\substack{\lambda \text{ is a} \\ (t,t+1)\text{-core}}} q^{\text{maj}(\lambda)} = \frac{1}{[t+1]_q} \begin{bmatrix} 2t \\ t \end{bmatrix}_q$$

Let λ be a s-c. $(2t, 2t+1)$-core.

Read off the levels of the defining beads of the corresponding abacus to give $b = (b_0, \ldots, b_t)$.

Define

$$\text{maj}(\lambda) = 2 \sum_{i\,:\,b_{i-1} \geq b_i} (2i - b_i - 1).$$

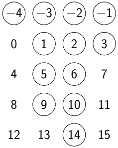
Then

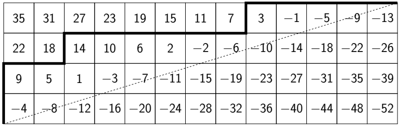
$$\sum_{\substack{\lambda \text{ is a self-conj.} \\ (2t,2t+1)\text{-core}}} q^{\text{maj}(\lambda)} = \begin{bmatrix} 2t \\ t \end{bmatrix}_{q^2}$$

Proof sketch

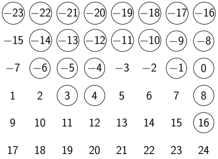
- Use Anderson's lattice path bijection:

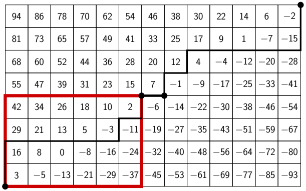
  (s,t)-flush abaci $\longleftrightarrow$ $L : (0,0) \to (s,t)$ above $y = \frac{t}{s}x$.





- Create a similar lattice path bijection: (improves Ford-Mai-Sze)
  antisymm. (s,t)-flush abaci $\longleftrightarrow$ $L : (0,0) \to \left(\lfloor \frac{s}{2} \rfloor, \lfloor \frac{t}{2} \rfloor\right)$.

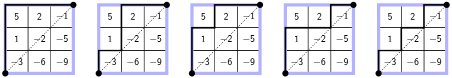




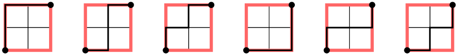
Proof sketch

- $(t, t+1)$-flush abaci $\longleftrightarrow$ $L : (0,0) \to (t,t)$ above $y = x$.

  Dyck paths!



- antisymm. $(2t, 2t+1)$-flush abaci $\longleftrightarrow$ $L : (0,0) \to (t,t)$.



- Use the major index on lattice paths that is known to give the desired q-analog: $$\text{maj}(L) = \sum_{i:(L_i, L_{i+1}) = (E,N)} i$$

$$q^0 + q^2 + q^3 + q^4 + q^{2+4} = \frac{1}{[4]_q} \begin{bmatrix} 6 \\ 3 \end{bmatrix}_q$$
$$q^0 + q^1 + q^2 + q^2 + q^3 + q^{1+3} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q$$

- Translate this major index to language of abaci and cores.

Talk Recap

- Definitions
  - Core partitions and abacus diagrams
  - Simultaneity
- Alcove geometry
  - Which alcoves are good representatives?
  - Simultaneous core partitions!
- Search for q-analogs of Catalan numbers
  - Piggy-back on lattice path combinatorics
  - A new major statistic on simultaneous cores.
- Remarkable
  - Type-independent setup.
  - Abaci are the right tool.

What's next?

1. **Core survey**
   - Compile combinatorial interpretations into illustrated dictionary.
   - Reconcile many appearances of cores into historical survey.
   - Gathering sources stage — What do you know?
2. **Open question: Catalan q-analogs**
   - **Question.** Is there a core statistic for m-Catalan $(t, tm \pm 1)$?
   - *Progress:* m-Catalan number C_3 through $(3, 3m+1)$-cores.
3. **Open question: Properties of simultaneous cores**
   - **Question.** What is the average size of an (s, t)-core partition?
   - *Progress:* Answer: $(s+t+1)(s-1)(t-1)/24$. Proof?
4. **Open question: Cyclic sieving phenomenon**
   - Note: $\frac{1}{[a+b]_q}\begin{bmatrix} a+b \\ a \end{bmatrix}_q\Big|_{q=-1} = \binom{\lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor}{\lfloor \frac{a}{2} \rfloor}$.

Thank you!

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Interact: `people.qc.cuny.edu/chanusa` $>$ `Animations`

Drew Armstrong, Christopher R. H. Hanusa, Brant C. Jones.
Results and conjectures on simultaneous core partitions.
Submitted, 2013. arXiv:1308.0572.

Christopher R. H. Hanusa and Brant C. Jones.
Abacus models for parabolic quotients of affine Coxeter groups
Journal of Algebra. Vol. 361, 134–162. (2012) arXiv:1105.5333

Gordon James and Adalbert Kerber.
The representation theory of the symmetric group.
Addison-Wesley, 1981.