A Gessel-Viennot Type Method for Cycle Systems

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Have title slide on projector.

I'm glad to have you all here today. I will step through my thesis highlighting the most important ideas and results. The title of this talk talks about counting cycle systems. We will talk about that in the second half of the talk.

Put "Outline" slide on projector.

I start with an introduction and brief history of the subject of the enumeration of domino tilings. Then I introduce the concept of Aztec pillows and generalized Aztec pillows. From there, I head into results, starting with matrix-theoretic results. I then present the Gessel-Viennot method, and show how I was able to extend it to a new method for counting tilings of generalized Aztec pillows. Lastly I touch on some future directions for study in this vein.

Without further ado, I present ... my thesis.

Put "Introduction" slide on projector.

So let's start at the beginning. The question of the day is "How many domino tilings are there of a given region?". The example you should have in mind is a chessboard that we want to cover by non-overlapping dominoes, or in other words 2×1 or 1×2 rectangles.

In the language of tilings, the chessboard will be a region, the rectangles are tiles, and any collection of tiles that covers the board is called a tiling.

So think in your head about how many domino tilings you might expect for a chessboard. How many people think that there are more than 10 possible tilings? Keep your hands up if you think there will be more than 100? 1,000? 10,000? 100,000? 1,000,000? 10,000,000?

Well, the correct answer is 12,988,816!

Fill in answer.

It may be surprising, but this number is a perfect square, equal to 3604^2. We've got an answer, but how are we better off for knowing this answer? This implies that we should actually change the question of the day to being "How can we calculate quickly how many domino tilings there are of a given region?". And how can we understand why this number is a perfect square?

Outline

- Introduction
- Kasteleyn-Percus Matrices
- Aztec Diamond-Type Regions
- Gessel-Viennot and the Hamburger Theorem
- Future Directions

Introduction

Consider a chessboard — "region"
and 32 non-overlapping unlabelled dominoes — "tiling"
covering the chessboard

— "tiles"

2×1 or 1×2
rectangles

Q: How

many domino tilings are there of a given region?

A: (chessboard)

$$\#R_{8\times 8} =$$

Introduction

Consider a chessboard — "region"
and 32 non-overlapping unlabelled dominoes covering the chessboard — "tiling"

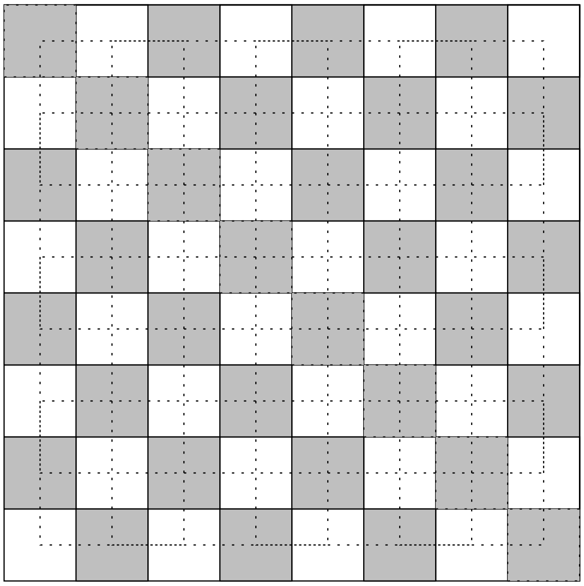
— "tiles"

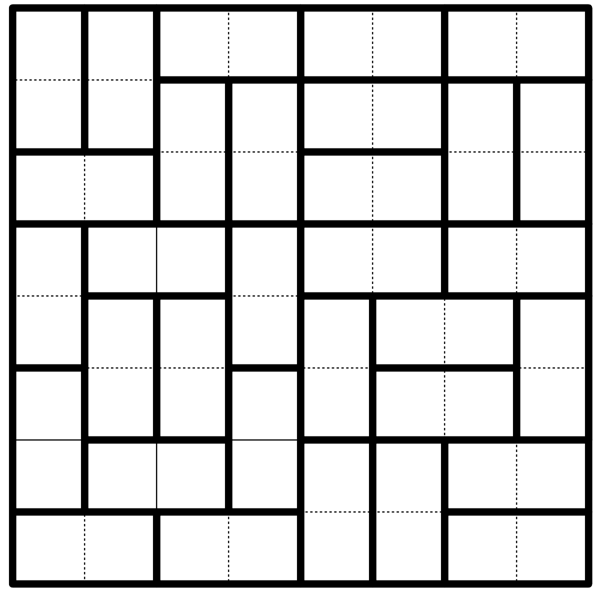
2×1 or 1×2
rectangles

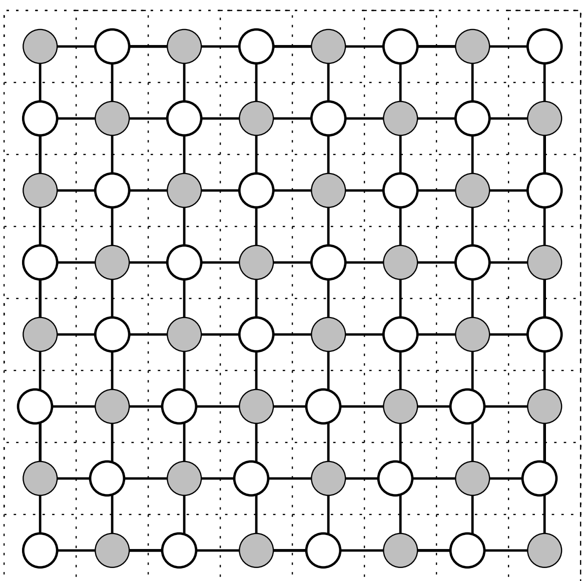
Q: How can we count quickly
how many domino tilings there are of a given region?

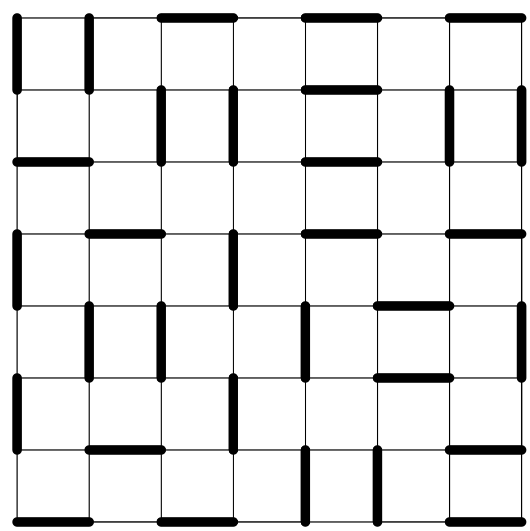
A: (chessboard)

$$\#R_{8\times 8} = 12{,}988{,}816$$









One way to get started is by considering a simple equivalence. Given any region, there is an associated dual graph to the region, as shown here in the case of a chessboard.

Use correspondence transparencies.

Remember that a perfect matching of a graph is a way to choose edges of the graph that cover completely the vertices of the graph.

Now given any domino tiling of the region, there is an associated perfect matching of the region's dual graph. If you see a vertical domino, you place a vertical edge on the graph, etc.

After now, when I say matching, I mean perfect matching.

So if we want to count the number of domino tilings of a region, we can just as easily count the number of matchings of the region's dual graph.

Write on blackboard domino tilings of region $\leftrightarrow$ *matching of region's dual graph.*

So now we want to know how we can count the number of matchings of a graph efficiently.

Put "History of the dual graph" slide on projector.

They popularized a way to count matchings using matrices in the case where the graph is bipartite and planar.

Remember that a bipartite graph is one where there are two types of vertices, black and white, such that the only edges are between black and white vertices. (There are no edges between vertices of the same color.)

Notice that there can only be a perfect matching if there are the same number of black and white vertices.

Given a bipartite graph, construct a matrix A with rows corresponding to black vertices and columns corresponding to white vertices. The entries $a_{i,j}$ of A are equal to 1 if there is an edge between black vertex b_i and white vertex w_j.

Recall that the permanent of a matrix is like the determinant of a matrix with no sign in the permutation expansion of the determinant. A non-zero term in the permanent of A corresponds exactly to finding n non-zero entries of A with one 1 in each row and each column. n such ones corresponds to n edges in the graph, covering each white vertex and each black vertex exactly once – AHA! a perfect matching.

Put "Matching Example" slide on projector.

This implies that we can take the permanent of this incidence matrix and we would get the number of matchings in the graph. This method was used by a physicist named Kasteleyn in the 1960's to calculate the number of matchings on a rectangular section of the square grid.

Then in 1963, another physicist named Percus found a condition on planar graphs that allowed for a sign convention to convert the permanent into a determinant.

You take this determinant and you get this formula. But somehow this isn't terribly satisfying. For one, it's not even clear that this formula produces an integer. Second, it doesn't quite tell you much combinatorial information about the answer. For example, when we are dealing with an $n \times n$ region, the formula gives either a square or two times a square. That number I gave on the previous slide $12,988,816 = (3604)^2$.

Notice that the size of this Kasteleyn-Percus determinant is of size $O(n^2)$, where n is the length of a side of the square. You only get a permanent on bipartite graphs. You can only convert it to a determinant on planar graphs.

What I am building up to is a method that allows us to calculate a determinant of size $O(n)$ on a certain type of graph that needs neither be bipartite nor planar.

Matrices from the dual graph

Kasteleyn-Percus

- On planar graphs
- On bipartite graphs

Evaluate a permanent

Evaluate a determinant

A: ($2m \times 2n$ chessboard)

$$\#R_{2m\times 2n} = \prod_{j=1}^{n}\prod_{k=1}^{m}\left(4\cos^2\frac{\pi j}{2n+1} + 4\cos^2\frac{\pi k}{2m+1}\right)$$

¡ Evaluate a determinant of size $O(n)$!

Example Matrix

$$\rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$\downarrow$	$\downarrow$	$\downarrow$
2 perfect matchings	permanent $= 2$	determinant $= 2$

We've been talking mostly about rectangular regions, but now let me introduce you to another nicer region.

Put "Aztec Diamonds" slide on projector.

This region is called an Aztec diamond. Aztec diamonds are a family of regions depending on the number of steps it has on its diagonals. It is a much nicer region since when we count its number of domino tilings it gives $2^{\binom{n+1}{2}}$.

This region was presented by physicists Grensing, Carlsen, and Zapp in the 1980's but they gave no indication of a proof. In 1992, Elkies, Kuperberg, Larsen, and Propp gave four different proofs of this formula.

Put "Aztec Pillows" slide on projector.

A related region that was introduced by Jim Propp in the 1990's. It is called an Aztec pillow because of its shape. Like the Aztec diamond it is rotationally symmetric. Unlike the Aztec diamond, its upper-left and lower-right border has steps of length 3 instead of length 1. We can index these pillows by the length of their middle $2 \times 2n$ belts. For example, this first pillow is $AP_?$ and the second one is $AP_?$. Notice that they have upper and lower plateaux that are of even length, and that the length of the plateaux are either 2 or 4 depending on the parity of n.

This region was interested because the number of tilings of the region appears to have a nice formula for its number of tilings. I'll talk more about this at the end of the talk.

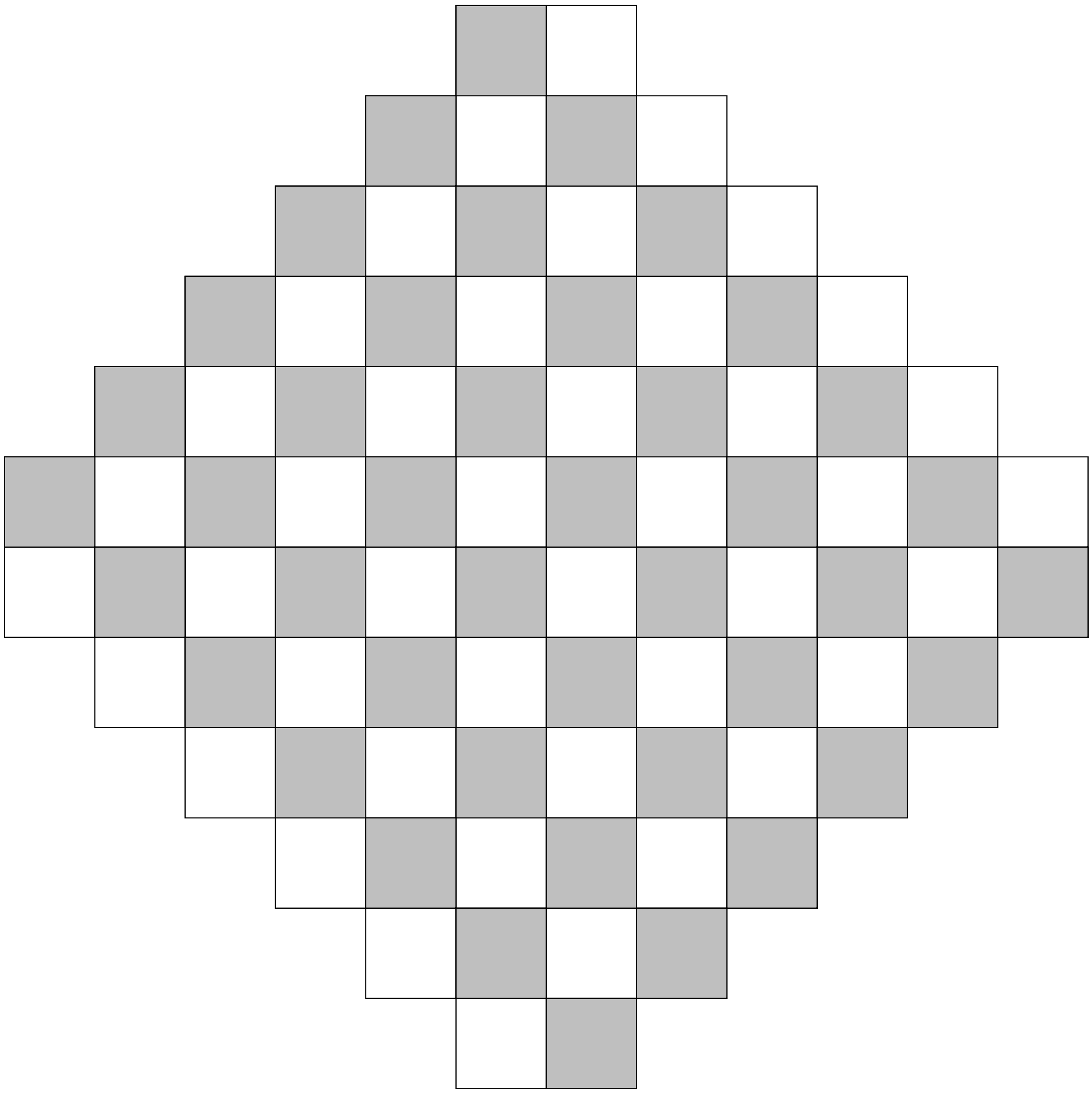
Notice that Aztec diamonds and Aztec pillows are rotationally symmetric about the origin and are embedded in the square grid. This implies that their number of tilings is a sum of two squares.

When working with Aztec pillows and experimental results that came with them, it became clear that we could generalize the notion of an Aztec pillow further.

Put "Generalized Aztec Pillows" slide on projector.

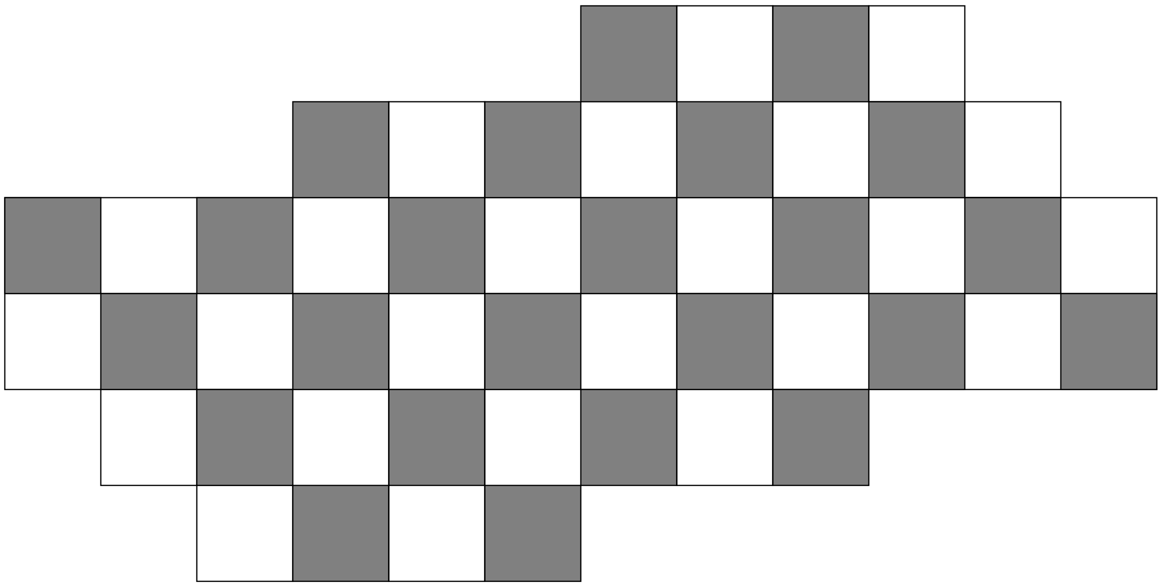
Just like with Aztec diamonds and Aztec pillows, we define a generalized Aztec pillow to be a region with steps that are of odd length (except the top and bottom plateaux which are even). They still have middle belts that are of even length. Here are a couple examples.

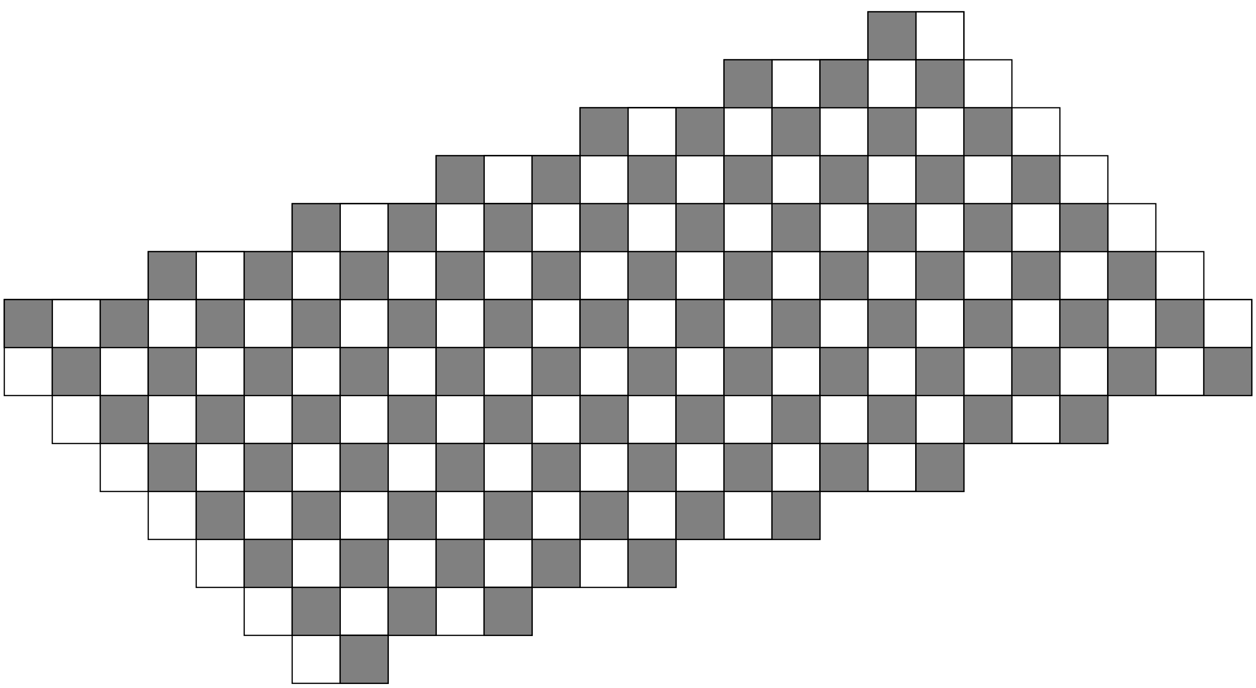
Aztec Diamonds



$$\#AD_n = 2^{n(n+1)/2}$$

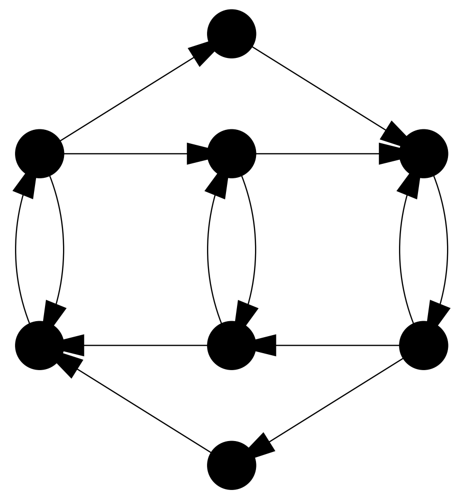
Aztec Pillows

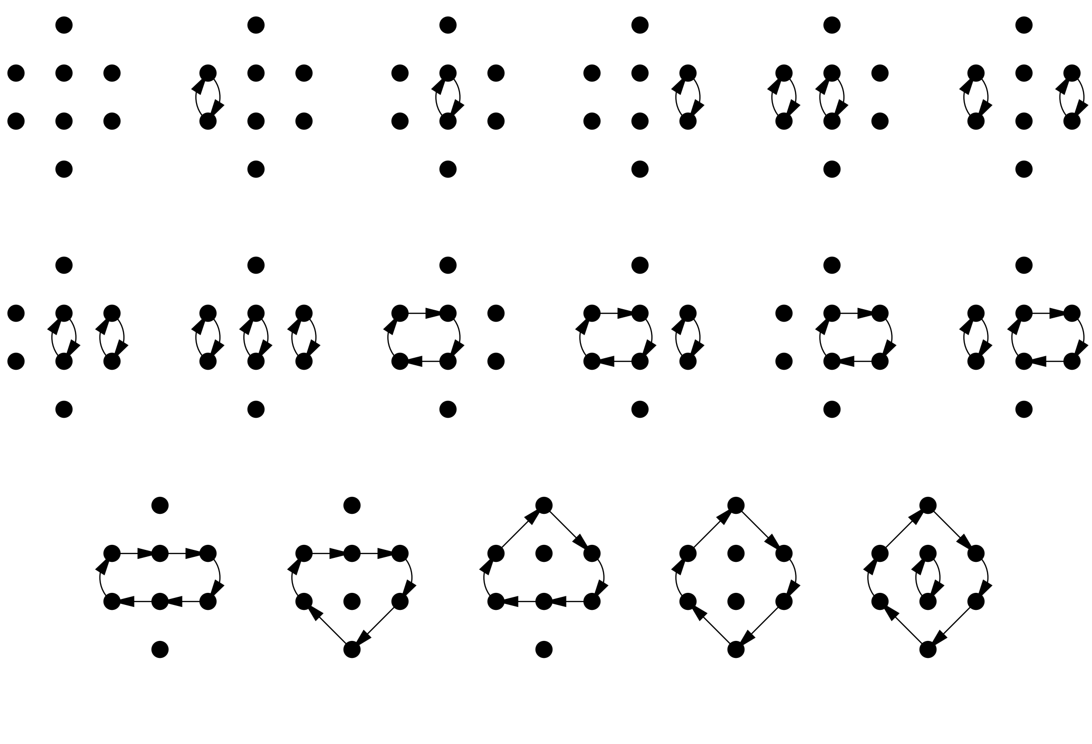


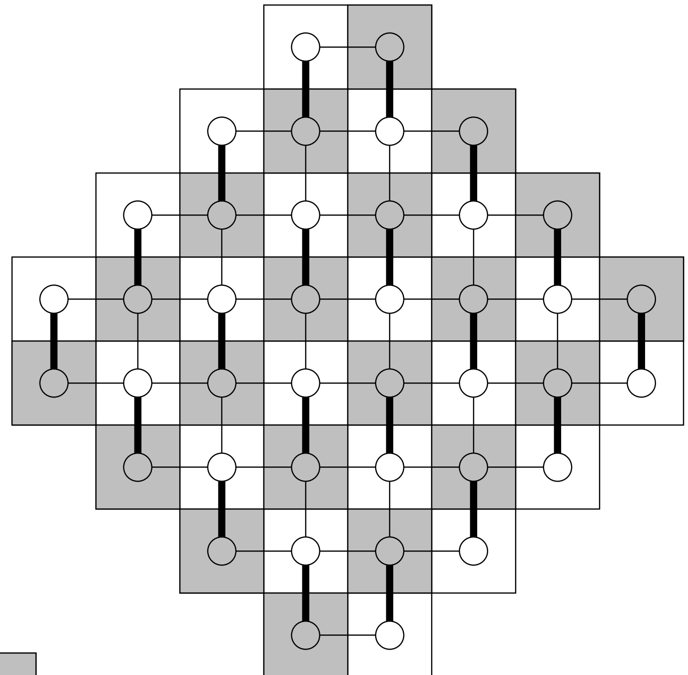


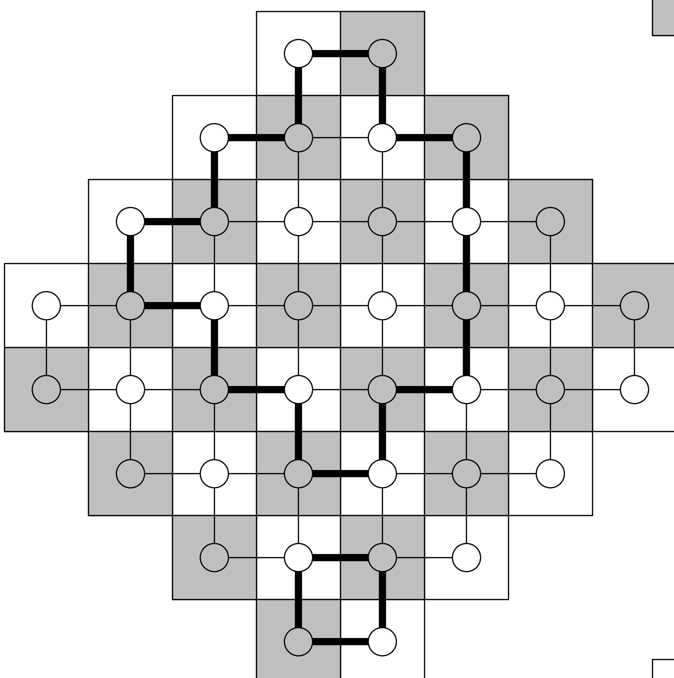
Cycle Systems

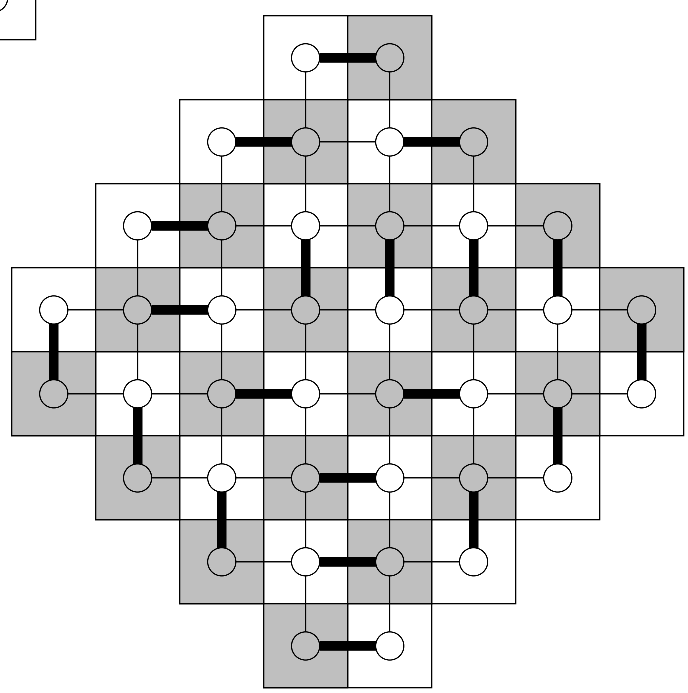
Given a directed graph G, a *cycle system* is a union of vertex-disjoint cycles.



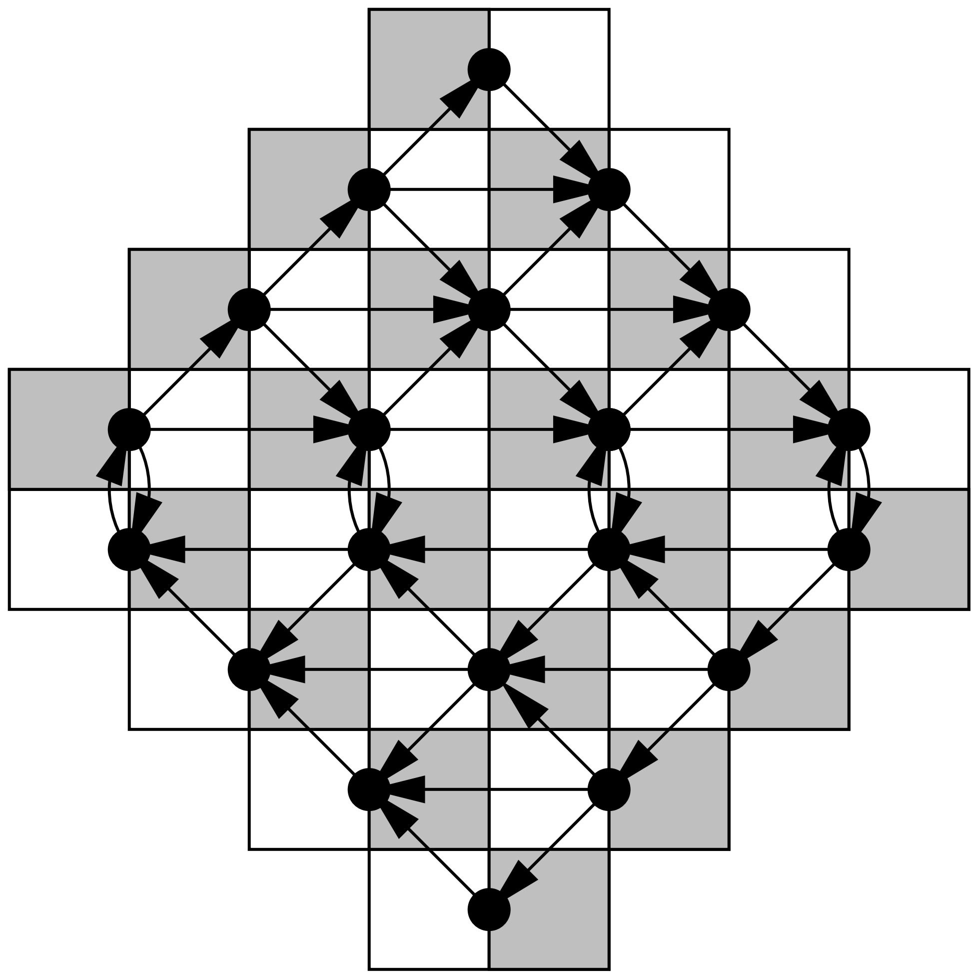








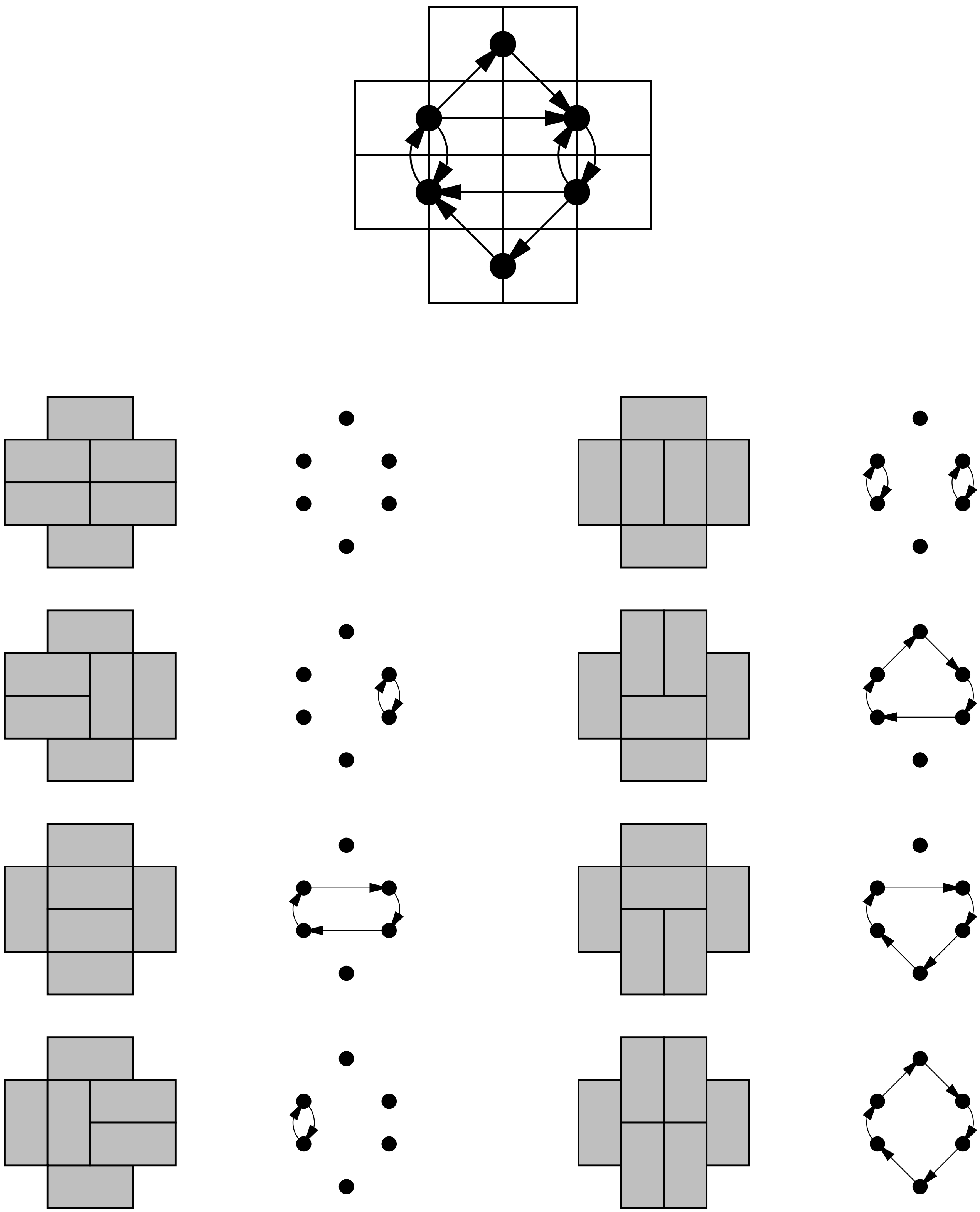
The digraph of an Aztec diamond



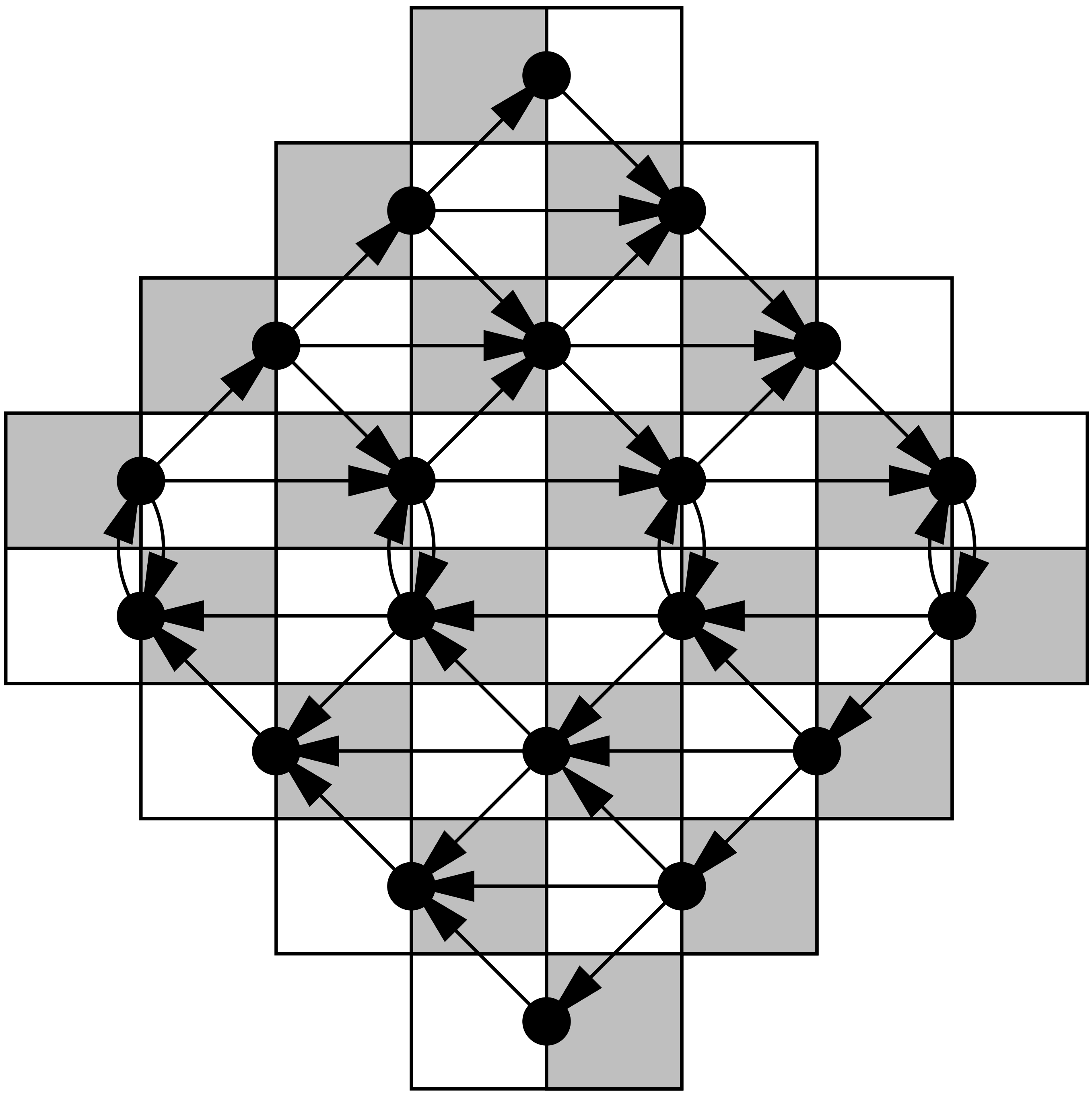
(Collapsing edges from the base matching)

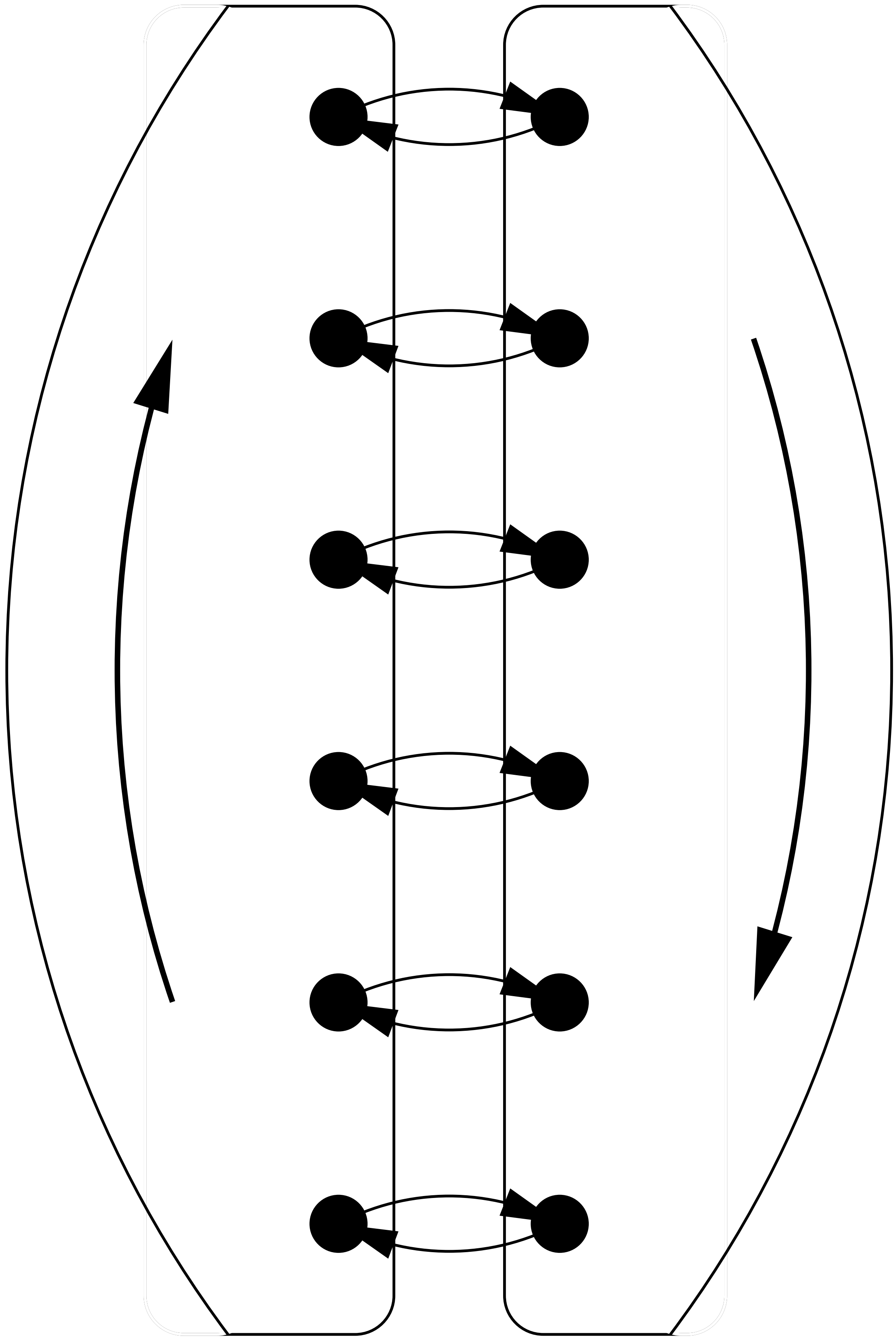
(Introduced by Brualdi and Kirkland)

Domino Tilings $\leftrightarrow$ Cycle Systems



A closer look reveals...





Now I want to go into another interesting approach to deal with the dual graph of a region.

Put "Natural Matching" slide on projector.

If we place some simple matching on the graph and then superpose an arbitrary matching on top, we know that taking the symmetric difference will yield a union of cycles. (Right? A 1-factor plus a 1-factor yields a 2-factor.)

Put "Directed Edges" slide on projector.

Now if we put directions on the edges so that in N, we send edges from black vertices to white vertices while in M we send edges from white vertices to black vertices, the resulting cycles when we superpose the two matchings will also be directed.

Put "cycles" slide on projector.

So there is a one to one correspondence between unions of directed cycles and matching of the dual graph. In particular, we are looking for directed cycles in this graph.

We can do one further simplification step. These blue edges are constant in each of the graphs, so we could easily contract them to a point, and the cycle system structure would stay the same.

Put "The digraph of an Aztec diamond" slide on projector.

This is what the resulting graph looks like in the case of an Aztec diamond. This graph was first presented by Brualdi and Kirkland in 2003. Given a region's dual graph, we call this the region's digraph.

In particular, this implies that the number of matchings of the dual graph is equal to the number of cycle systems in the region's digraph.

Write on blackboard <-> cycle systems in the region's digraph.

Quick Detour to the 1980's

Gessel and Viennot

Lindström

Karlin and McGregor

Count non-intersecting path systems

A path system $\mathcal{P}$

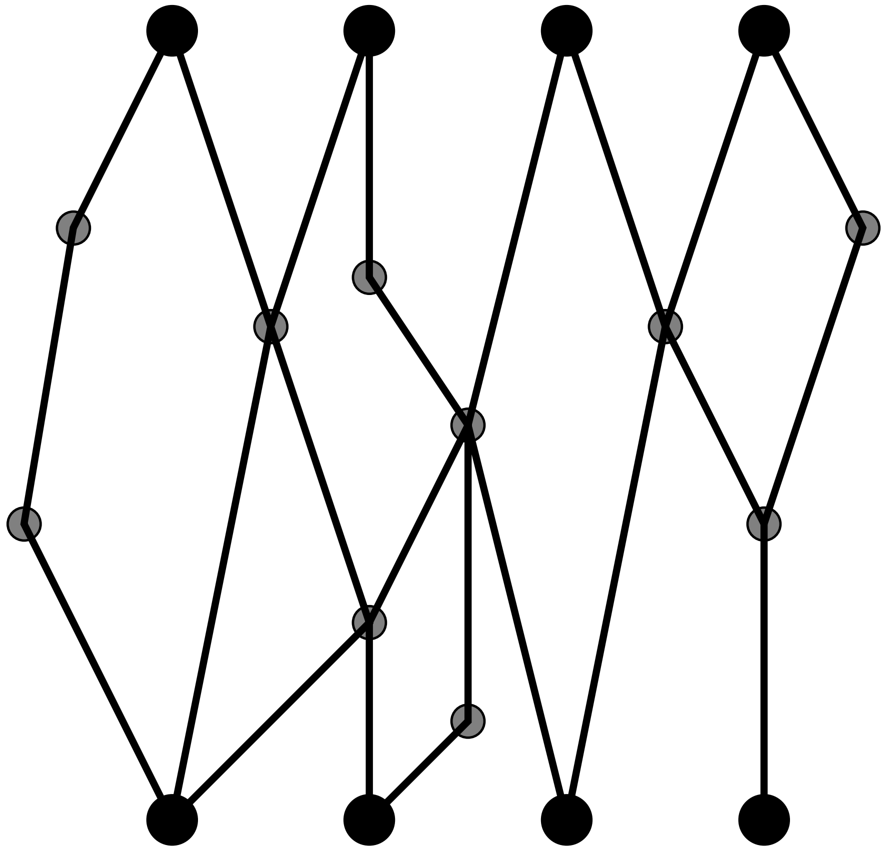
- k vertex-disjoint paths $q_i : s_i \to t_{\sigma(i)}$
- a permutation $\sigma \in S_k$

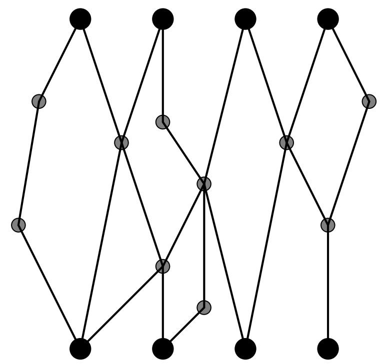
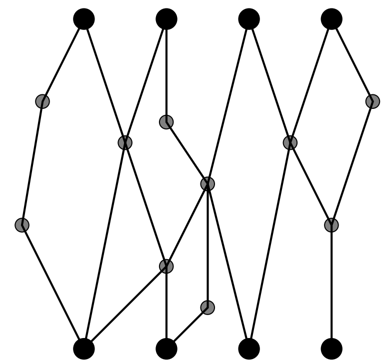
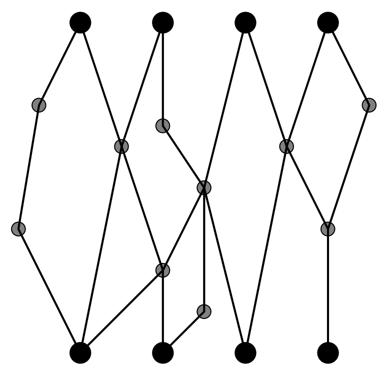
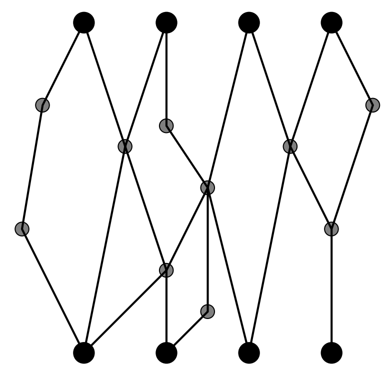
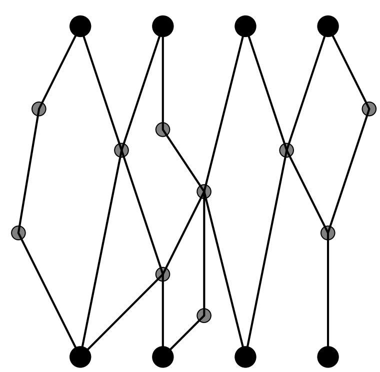
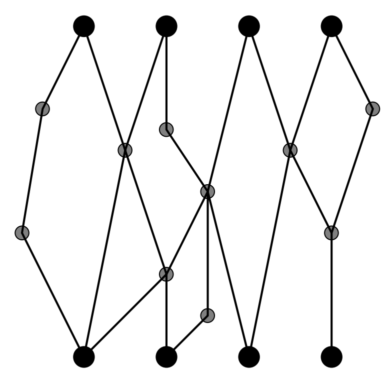
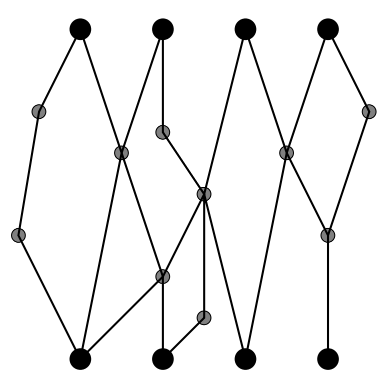
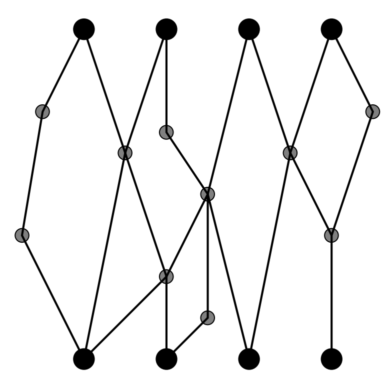
The sign of $\mathcal{P} = \mathrm{sgn}\,(\sigma)$

$p^+ = \#\,\mathcal{P}$ such that $\mathrm{sgn}\,(P) = +1$
$p^- = \#\,\mathcal{P}$ such that $\mathrm{sgn}\,(P) = -1$

The matrix $A = (a_{ij})$; $a_{ij} = \#$paths from $s_i \to t_j$

Theorem (G-V, 1985–89): $\det A = p^+ - p^-$.



Idea of Proof: Gessel-Viennot

Permutation Decomposition of the Determinant

$$\det A = \sum_{\sigma \in S_k} \mathrm{sgn}\,(\sigma)\, a_{1\sigma(1)} \cdots a_{k\sigma(k)}$$

- If $\mathcal{P} = \{q_1, \ldots, q_k\}$ is non-intersecting
  - Contributes $\mathrm{sgn}\,(\sigma)$ to $\det A$
- If $\mathcal{P} = \{q_1, \ldots, q_k\}$ has intersecting paths
  - Sign-reversing involution $\Rightarrow$ Contributes 0.

Count non-intersecting cycle systems

A cycle system $\mathcal{C}$

- A union of vertex-disjoint directed cycles

The sign of $\mathcal{C} = (-1)^{\ell+m}$

$\ell = \#$ of edges from G_2 to G_1 in $\mathcal{C}$.
$m = \#$ of vertex-disjoint cycles in $\mathcal{C}$.

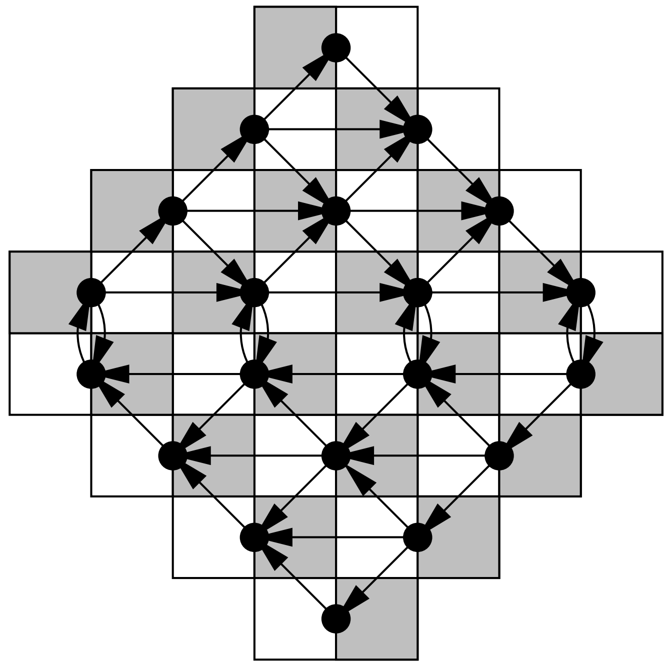
$c^+ = \# \mathcal{C}$ such that $\text{sgn}\,(C) = +1$
$c^- = \# \mathcal{C}$ such that $\text{sgn}\,(C) = -1$

The matrix

$$M_H = \begin{bmatrix} A & I_k \\ -I_k & B \end{bmatrix} \qquad A = \left\{ \begin{array}{c} \#\text{ paths from} \\ v_i \text{ to } v_j \text{ in } G_1 \end{array} \right\} \qquad B = \left\{ \begin{array}{c} \#\text{ paths from} \\ w_{k+i} \text{ to } w_{k+j} \text{ in } G_2 \end{array} \right\}$$

Theorem (H, 2004): $\det M_H = c^+ - c^-$.
(The "Hamburger Theorem")

Example: AD_4



$$M_H = \left[\begin{array}{cccc|cccc} 1 & 2 & 6 & 22 & & & & \\ & 1 & 2 & 6 & & & & \\ & & 1 & 2 & & & & \\ & & & 1 & & & & \\ \hline & & & & 1 & & & \\ & & & & 2 & 1 & & \\ & & & & 6 & 2 & 1 & \\ & & & & 22 & 6 & 2 & 1 \end{array}\right]$$

$1, 2, 6, 22, 90, 394, \ldots$
large Schröder N°s
A006318
. . .combin. interp.. . .

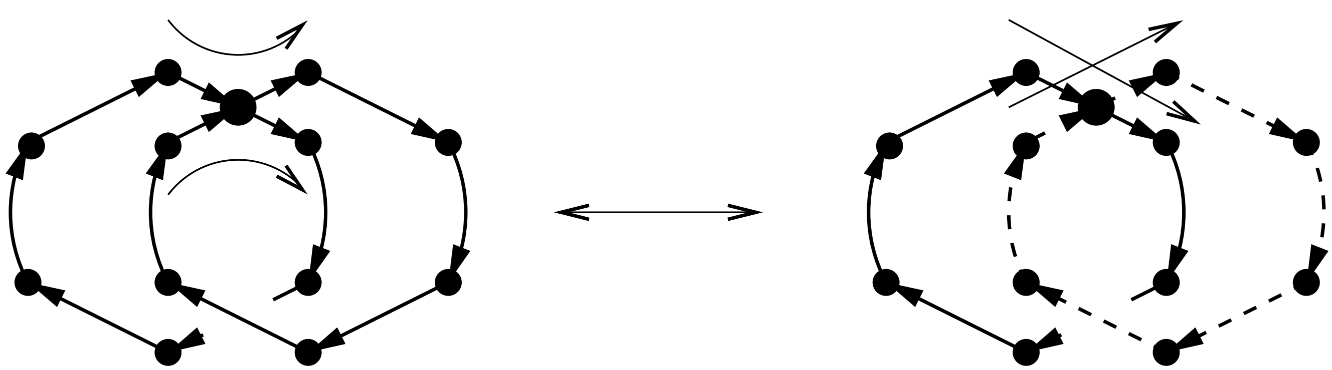
$$\det M_H = 2^{10} = \#AD_4$$

Idea of Proof

Key: Terms cancel in the permutation decomposition of the determinant of M_H.

Definition: A *walk system* is a union of directed cycles which may not be vertex-disjoint.

Lemma 1. If a walk system $\mathcal{W}$ contains a walk that is self-intersecting or contains two intersecting walks that are not 2-cycles, $\mathcal{W}$ belongs to one well-defined family $\mathcal{F}$ of walk systems of these types that cancel each other in the permutation expansion of the determinant of M_H.

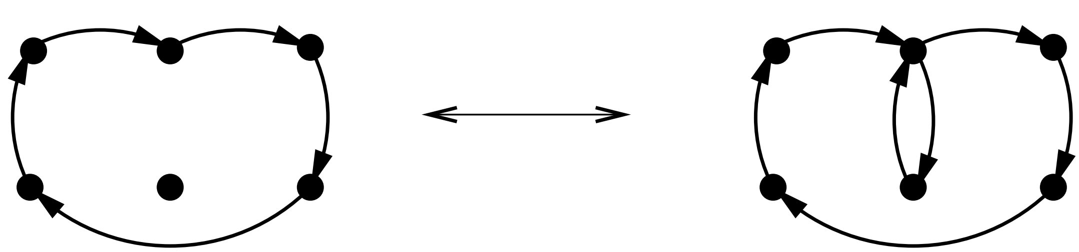


Idea of Proof

Definition: A permutation σ is *minimal* if it is a product of disjoint cycles of the form $(i_0, k + i_0)$ or $(i_0, i_1, k + i_1, k + i_2, i_2, \ldots, k + i_\ell, k + i_0)$.

Definition: A *minimal walk system* is a walk system such that its corresponding permutation is minimal. (=Original Def'n of Cycle System)

Lemma 2. Let $\mathcal{W}$ be a walk system that does not satisfy the conditions of Lemma 1. If $\mathcal{W}$ is not minimal or if it contains a walk that intersects with a 2-cycle, $\mathcal{W}$ belongs to one well-defined family $\mathcal{F}$ of walk systems of these types that cancel each other in the permutation expansion of the determinant of M_H.



(12364) (1364)(25)

Idea of Proof

Lemma 3. The walk system $\mathcal{W}$ is a minimal cycle system with vertex-disjoint cycles if and only if $\mathcal{W}$ does not satisfy the conditions of Lemmas 1 and 2.

Therefore only minimal cycle systems contribute (their sign) to the determinantal expansion of M_H.

Hence, $\det M_H = c^+ - c^-$ •

Why is this exciting?

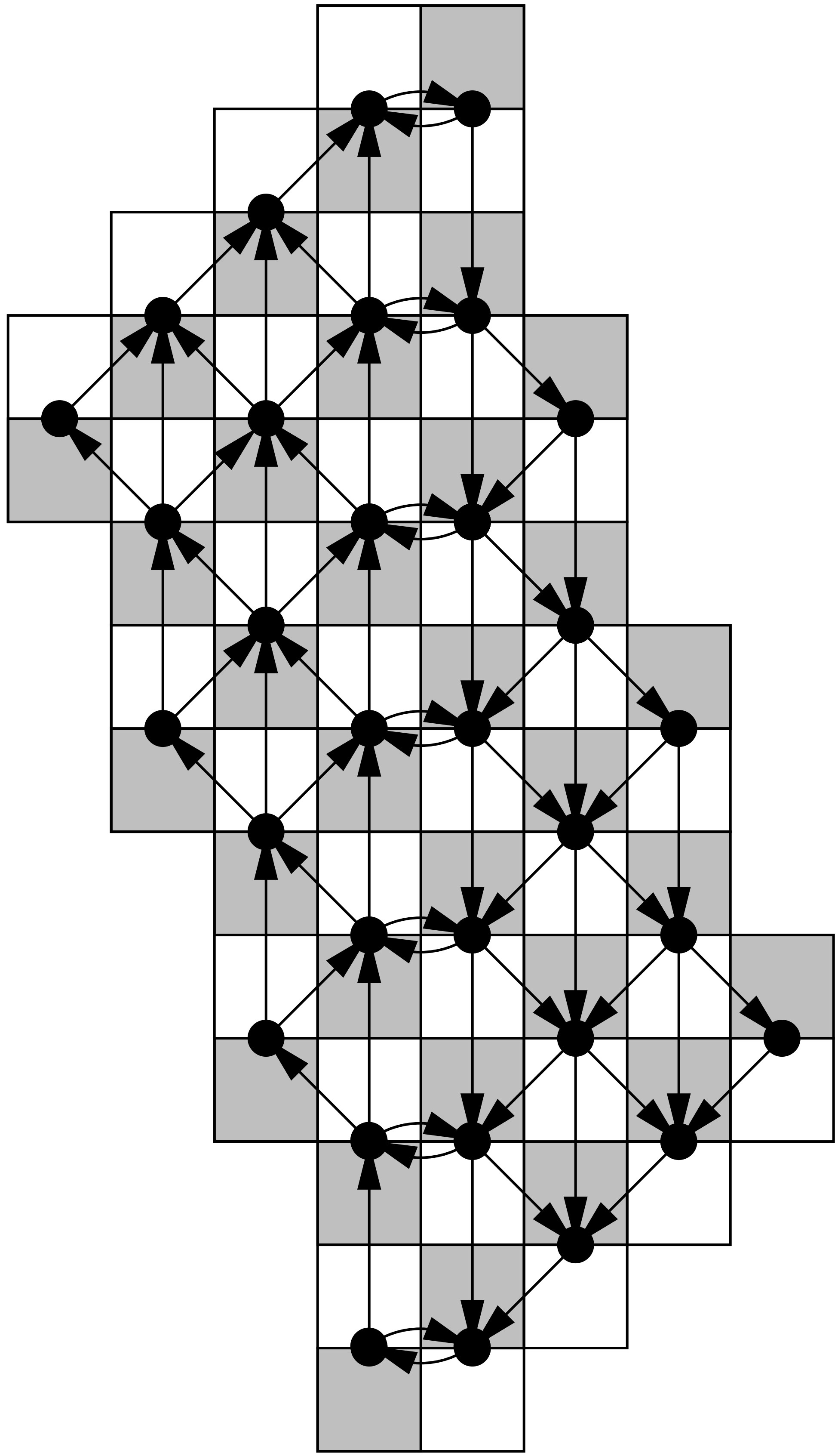
- It's pretty
- Works on non-planar graphs
- Reduces determinant size
  - on AD_n: $n(n-1) \to 2n$
  - Schur complement: $2n \to n$
- Purely combinatorial
- Applies to generalized Aztec pillows

Schur Complement

What's next?

- Generalization of Propp's Conjecture

- Relaxation of hamburger structure?



Propp's Conjecture and Generalizations

Conjecture (Propp, 1999):

$$\#AP_{2n+1} = \ell_n^2 s_n \qquad \ell_n = ?$$

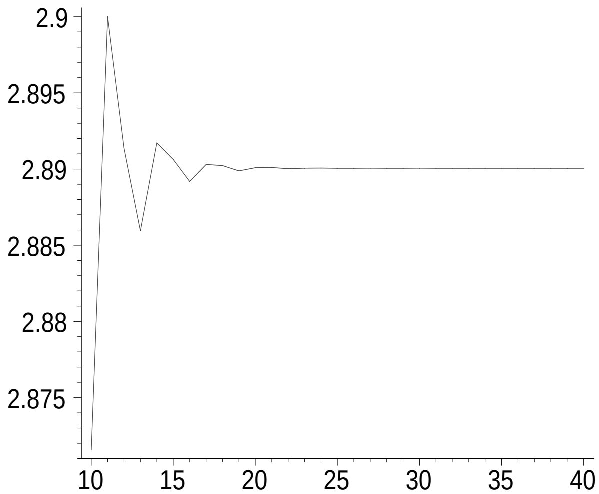
$$\sum_{n=0}^{\infty} s_n x^n = \frac{5+3x+x^2-x^3}{1-2x-2x^2-2x^3+x^4}$$

$$\#AP_{2n+2} = \ell_n^2 s_n \qquad \ell_n = ?$$

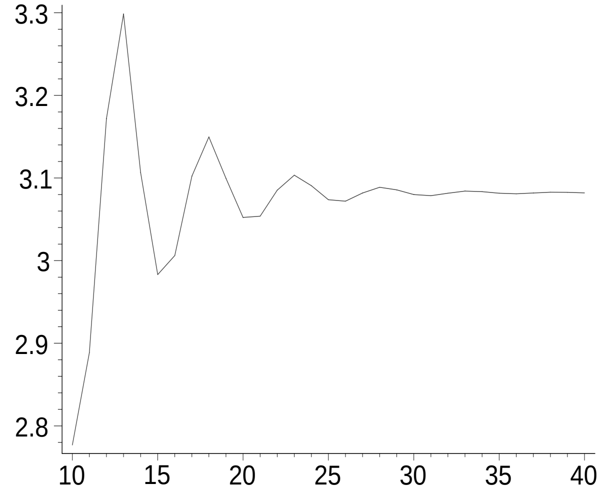
$$\sum_{n=0}^{\infty} s_n x^n = \frac{5+6x+3x^2-2x^3}{1-2x-2x^2-2x^3+x^4}$$

Already known	New Conjecture
$\#AP_n = a^2 + b^2$	$\ell_n \mid a$, $\ell_n \mid b$
	ℓ_n divides minors of hamburger matrix
	ℓ_n divides $\#(AP \setminus \{\text{tiles}\})$ (and not ℓ_n^2)
	odd pillows also have $\ell_n^2 s_n$ (no recurrence yet for s_n).
	Let $p(n) = \#AP_{n-3}$. Then $p(m) \mid p(n)$ when $m \mid n$.

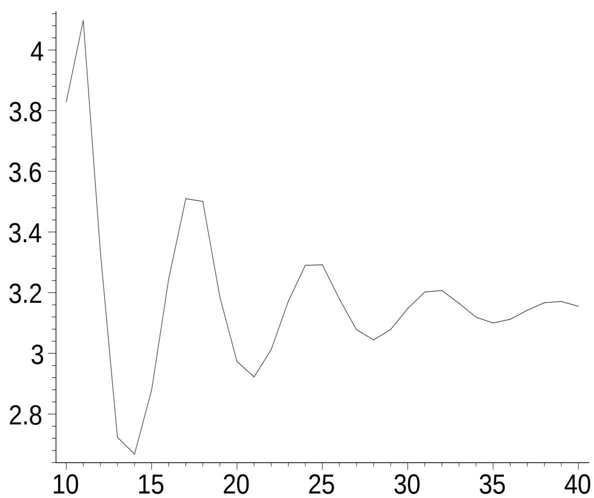
Damped Sinusoidal Behavior



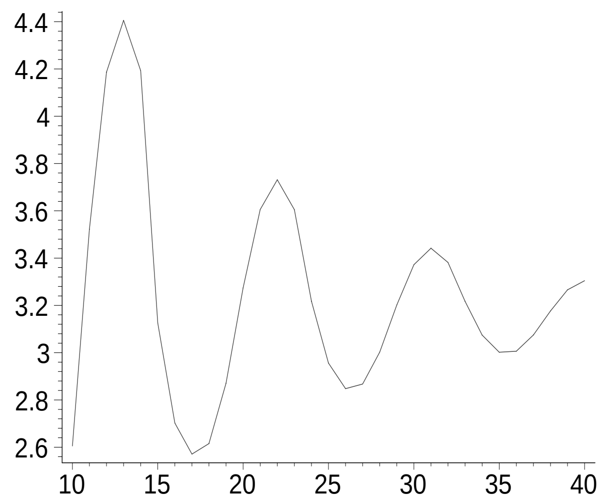
s_n/s_{n-2} for $\#AP_n^3$



s_n/s_{n-2} for $\#AP_n^5$



s_n/s_{n-2} for $\#AP_n^7$



s_n/s_{n-2} for $\#AP_n^9$

Data

n	$\#AP_n^3$	s_n
1	$\mathbf{2}$	2
2	$\mathbf{5}$	5
3	$2^2 \cdot \mathbf{5}$	5
4	$3^2 \cdot \mathbf{13}$	13
5	$\mathbf{2}^{10}$	16
6	$19^2 \cdot \mathbf{37}$	37
7	$2^4 \cdot \mathbf{3^2} \cdot \mathbf{5} \cdot 19^2$	45
8	$\mathbf{109} \cdot 263^2$	109
9	$\mathbf{2}^9 \cdot 3^4 \cdot \mathbf{5} \cdot 11^2 \cdot \mathbf{13}$	130
10	$3^4 \cdot \mathbf{313} \cdot 911^2$	313
11	$2^6 \cdot 3^2 \cdot \mathbf{13} \cdot \mathbf{29} \cdot 43^2 \cdot 71^2$	377
12	$\mathbf{5} \cdot 11^4 \cdot 31^2 \cdot 151^2 \cdot \mathbf{181}$	905
13	$\mathbf{2}^{28} \cdot 7^2 \cdot \mathbf{17}^3 \cdot 31^2$	1088
14	$101^2 \cdot 103^2 \cdot \mathbf{2617} \cdot 8363^2$	2617
15	$2^8 \cdot \mathbf{5} \cdot \mathbf{17}^3 \cdot 19^2 \cdot \mathbf{37} \cdot 53^2 \cdot 71^2 \cdot 89^2$	3145
16	$31^2 \cdot \mathbf{7561} \cdot 27283^2 \cdot 35149^2$	7561
17	$\mathbf{2}^{17} \cdot \mathbf{3^2} \cdot \mathbf{5} \cdot 11^2 \cdot 19^4 \cdot 59^2 \cdot 61^2 \cdot \mathbf{101} \cdot 241^2$	9090
18	$3^{10} \cdot 5^2 \cdot \mathbf{13} \cdot 29^2 \cdot \mathbf{41}^4 \cdot 43^2 \cdot 211^2 \cdot 1723^2$	21853
19	$2^{10} \cdot 23^2 \cdot 43^2 \cdot \mathbf{109} \cdot \mathbf{241} \cdot 263^2 \cdot 439^2 \cdot 461^2 \cdot 593^2$	26269

Thanks!

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Thank you

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