Let's Count!
Enumeration through Matrix Methods

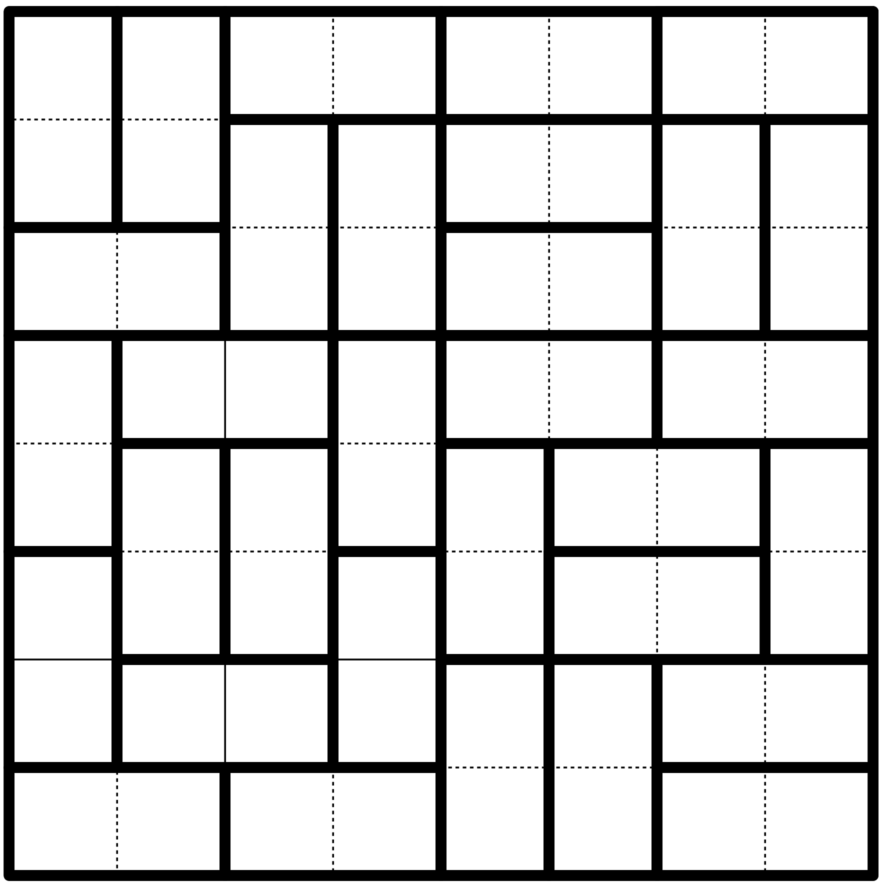
Christopher Hanusa

Queens College

February 14, 2008

Counting

Example: How many ways are there to place 32 blank dominoes on a chessboard?



Answer:

Outline

- Enumeration of disjoint path systems
  - Introduction to path systems
  - The Gessel–Viennot method
  - Applications of Gessel–Viennot
- Enumeration of domino tilings
  - Perfect matchings of a graph
  - Kasteleyn–Percus matrices
- Open Problems

Path Systems

A <u>graph</u> $G = (V, E)$.

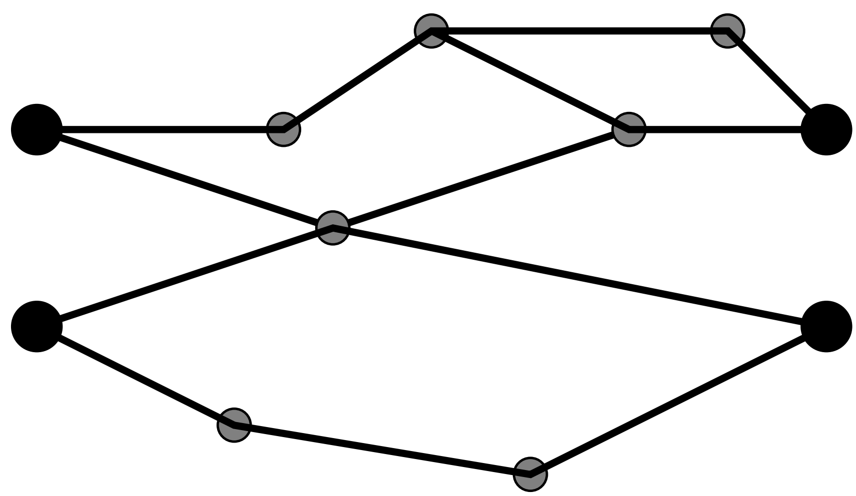
A <u>directed graph</u>: Orient each edge $e \in E$.

A <u>path</u> from vertex a to vertex b:

Two paths are <u>disjoint</u>: They share no vertices.

Path Systems

A path system $\mathcal{P}$ from $\mathcal{A}$ to $\mathcal{B}$ in G:



Recall: $\text{sign}(\sigma) = (-1)^{\#\text{ of transp. that compose }\sigma}$

Define: $\text{sign}(\mathcal{P}) = \text{sign}(\sigma)$.

Combinatorial data:
Define $m_{ij} = \#$ of paths from a_i to b_j in G.

$$M = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

The Gessel–Viennot Method

Let G be a directed graph.

Let $\mathcal{A} = \{a_1, a_2, \ldots, a_k\} \subseteq V(G)$.

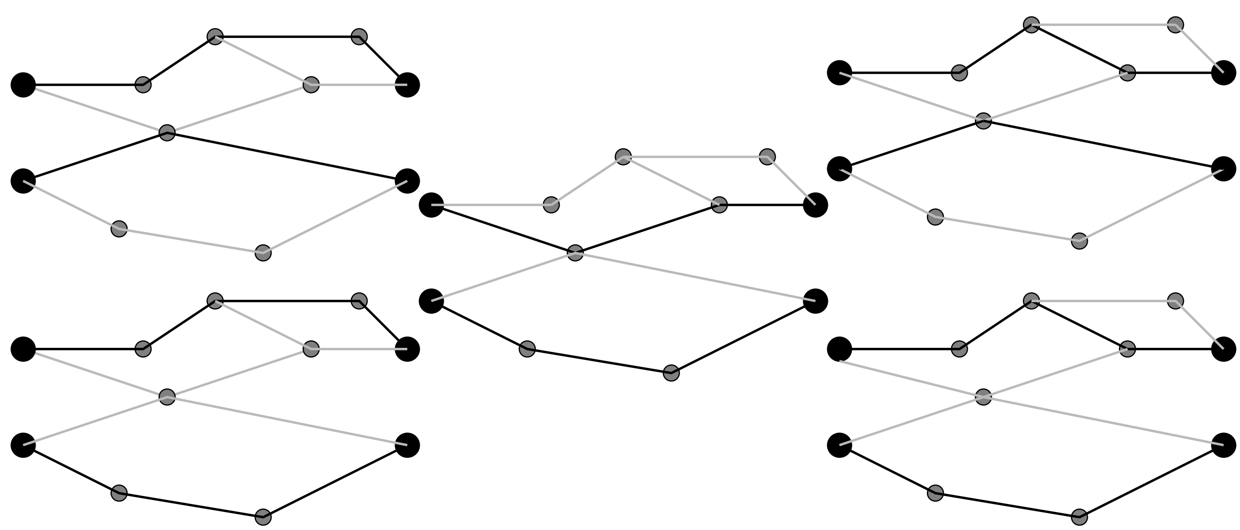
Let $\mathcal{B} = \{b_1, b_2, \ldots, b_k\} \subseteq V(G)$.

Define $M = (m_{ij})_{1 \leq i,j \leq k}$.

Then

$$\det M \;=\; \sum_{\substack{\text{vertex-}\\ \text{disjoint}\\ \mathcal{P}}} \text{sign}(\mathcal{P})$$

In our example, $\det M = 5$.



Consequences

①

A combinatorial question of counting path systems can be evaluated using a determinant.

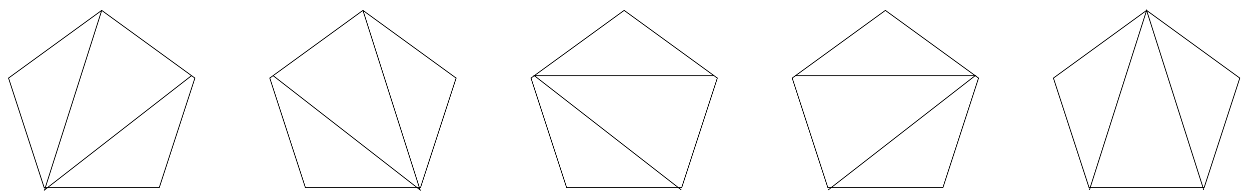
②

A determinant may be evaluated by counting path systems in an associated lattice.

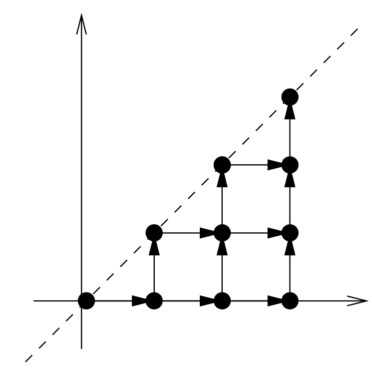
Example: A Catalan Determinant

	$c_0,$	$c_1,$	$c_2,$	$c_3,$	$c_4,$	$c_5,$	$c_6,$	$c_7,$	$\ldots$
Catalan Numbers:	1,	1,	2,	5,	14,	42,	132,	429,	$\ldots$

Interpretation: Triangulations of an $(n+2)$-gon.



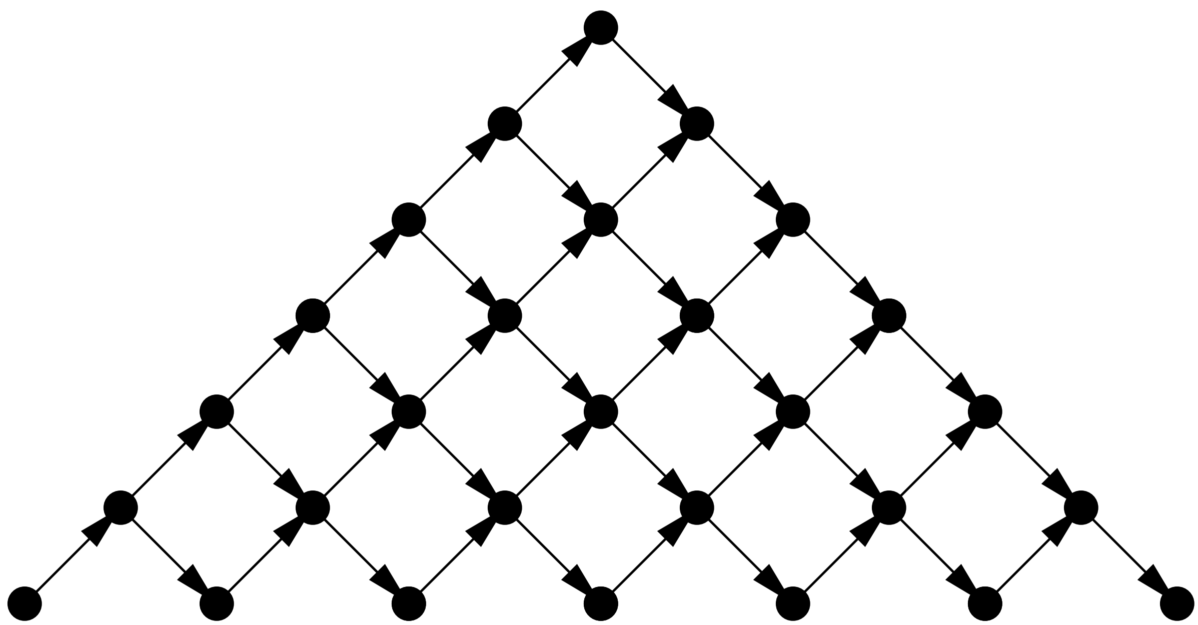
Also: Number of lattice paths from $(0,0)$ to (i,i):



Did you know?

$$\det \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_1 & c_2 & & & \\ c_2 & & & & \vdots \\ \vdots & & & & \\ c_n & & \cdots & & c_{2n} \end{pmatrix} = 1$$

Example: A Catalan Determinant



\# of paths from a_i to b_j $= c_{i+j}$.

$\det M =$ # disjoint path systems from $\mathcal{A}$ to $\mathcal{B}$ in the above lattice

Path systems disjoint $\Rightarrow$ paths must be $a_0 \to b_0$, $a_1 \to b_1$, $a_2 \to b_2$, $\ldots$, $a_n \to b_n$.

There is only one possibility, so $\det M = +1$

Proof of Gessel–Viennot

$$\sum_{\substack{\text{all path}\\ \text{systems } \mathcal{P}}} \text{sign}(P) = \sum_{\substack{\text{all}\\ \sigma \in S_k}} \text{sign}(\sigma) \binom{\#\text{ path systems}}{\text{w/ perm. } \sigma}$$

$$= \sum_{\substack{\text{all}\\ \sigma \in S_k}} \text{sign}(\sigma)\, m_{1\sigma(1)} m_{2\sigma(2)} \cdots m_{k\sigma(k)}$$

$$= \det M$$

In order to prove

$$\sum_{\substack{\text{disjoint}\\ \mathcal{P}}} \text{sign}(P) \;=\; \det M,$$

we need to prove

$$\sum_{\substack{\text{non-disjoint}\\ \mathcal{P}}} \text{sign}(P) \;=\; 0.$$

Proof of Gessel–Viennot

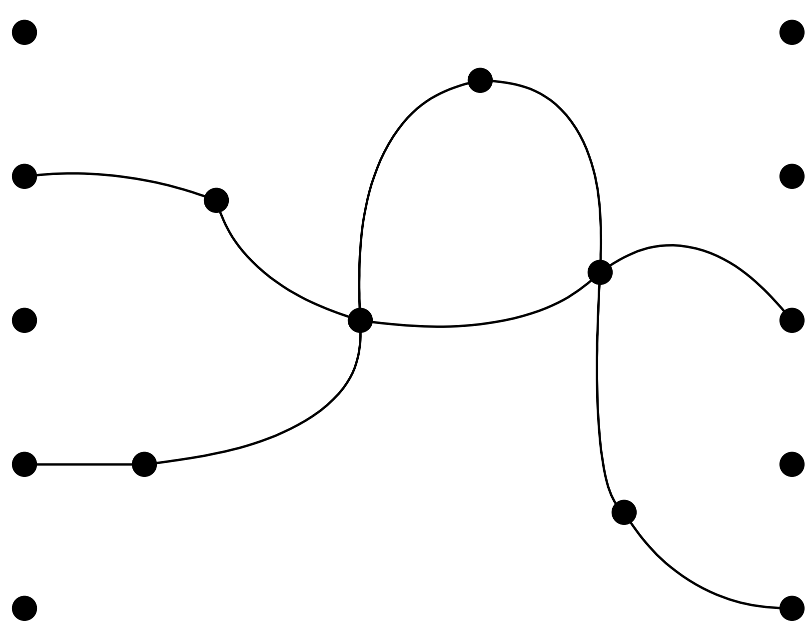
We want to prove $\sum_{\substack{\text{non-disjoint} \\ \mathcal{P}}} \text{sign}(P) = 0.$

Define $N :=$ set of all non-disjoint $\mathcal{P}$.

We will construct an involution $\pi : N \to N$.

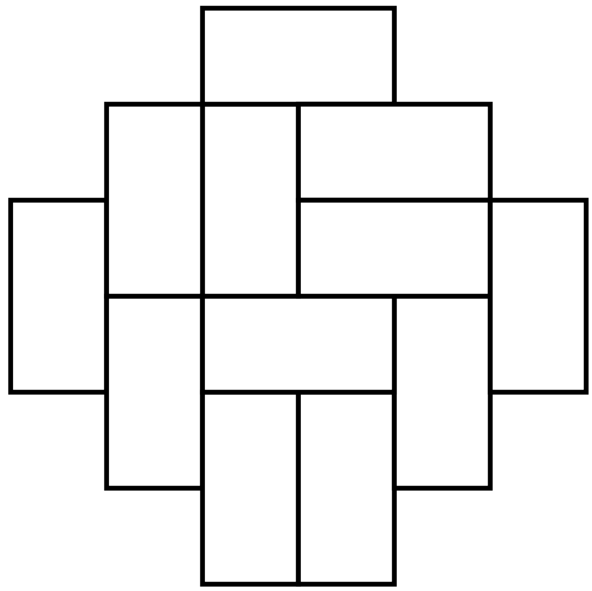
- $\pi^2 = \text{id}_N$.
- $\pi : \begin{array}{ccc} {}^{+}\mathcal{P} & \mapsto & {}^{-}\mathcal{P} \\ {}^{-}\mathcal{P} & \mapsto & {}^{+}\mathcal{P} \end{array}$

π is simple:

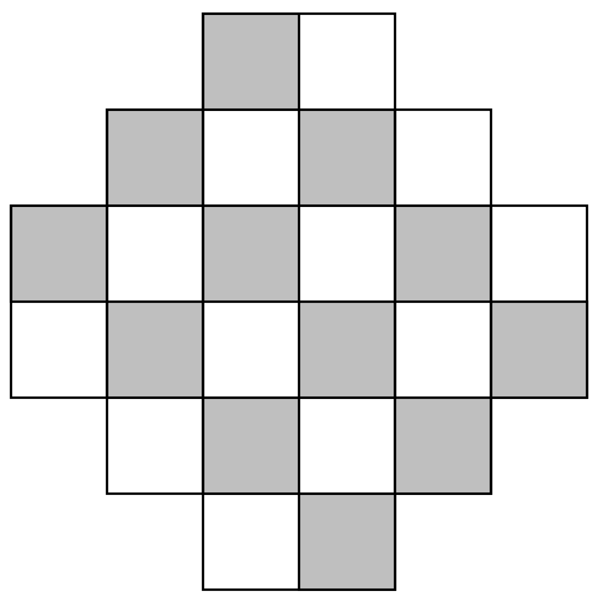


Application of Gessel–Viennot

Domino tiling

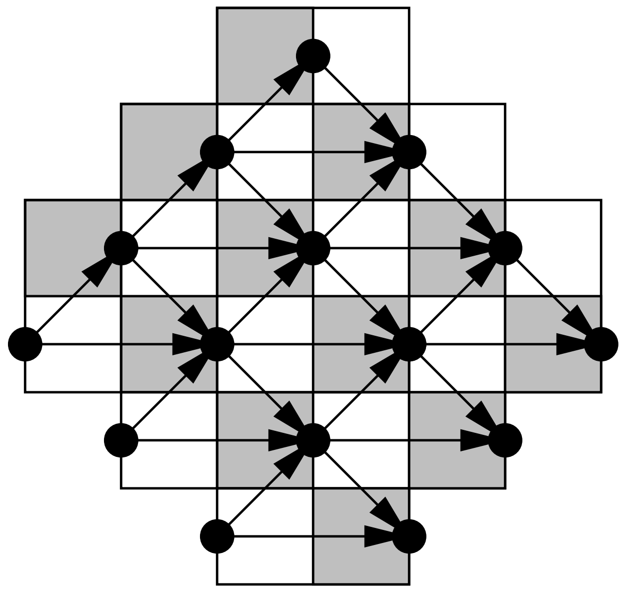


Aztec diamond



$$\#\,\mathrm{AD}_n = 2^{n(n+1)/2}$$

Domino Tiling $\longleftrightarrow$ Path System



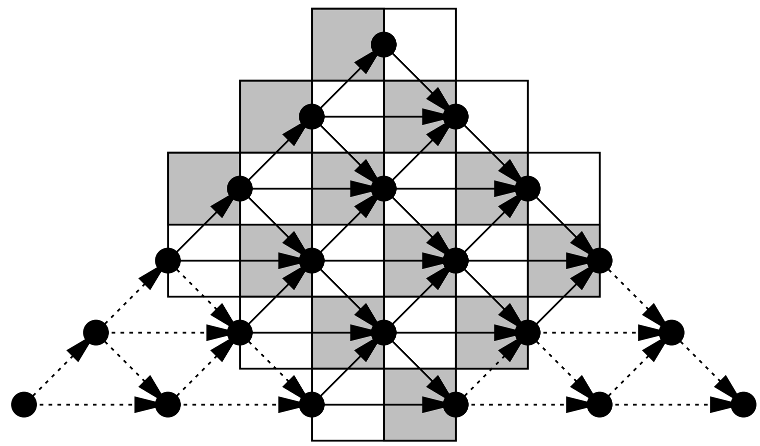
Start on the left. Traverse each domino directly through its center.

Domino Tiling

Path System

Place a domino following each path. The remaining dominoes are forced.

Counting Path Systems



$s_0, s_1, s_2, s_3, s_4, s_5, s_6, \ldots$

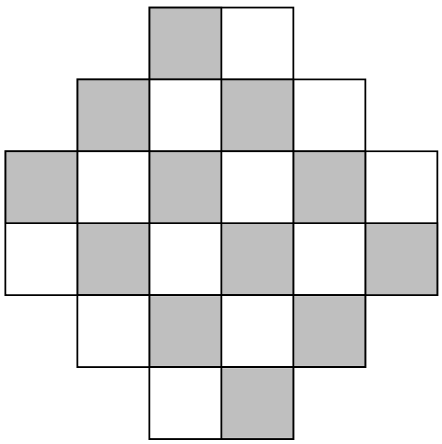
Large Schröder numbers: 1, 2, 6, 22, 90, 394, 1806, . . .
count the lattice paths from $(0,0)$ to (i,i):

$$\begin{aligned}
\#\text{ domino tilings} &= \#\text{ path systems} \\
&= \det \begin{pmatrix} 2 & 6 & 22 \\ 6 & 22 & 90 \\ 22 & 90 & 394 \end{pmatrix} \\
&= 2^6 \qquad (2^{n(n+1)/2} \text{ in general})
\end{aligned}$$

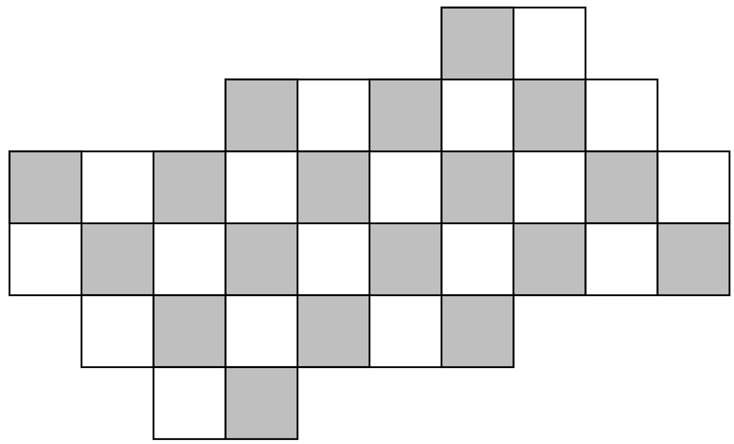
Method applies to generalized Aztec pillows.

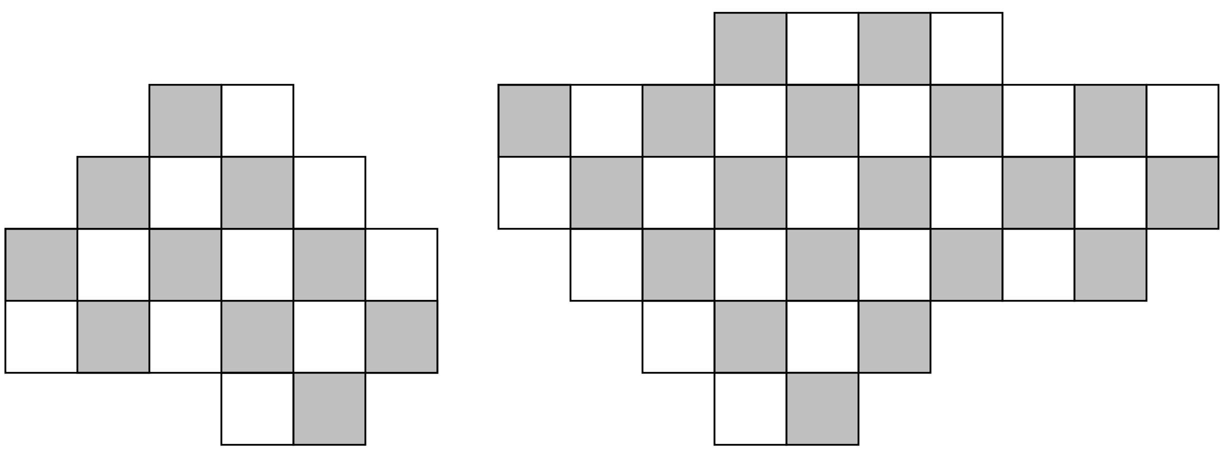
Aztec Regions

Aztec diamond



Aztec pillow



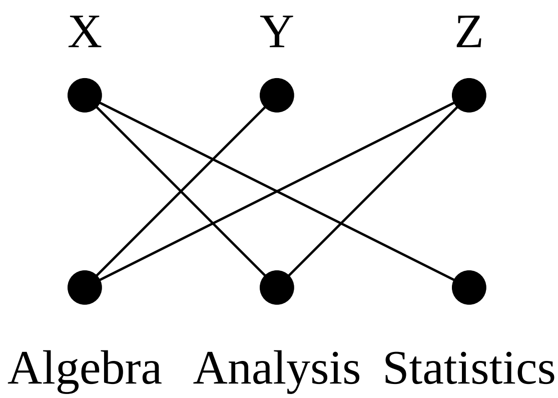


generalized Aztec pillows

Perfect Matchings

A (perfect) matching is a selection of edges that pairs all the vertices.

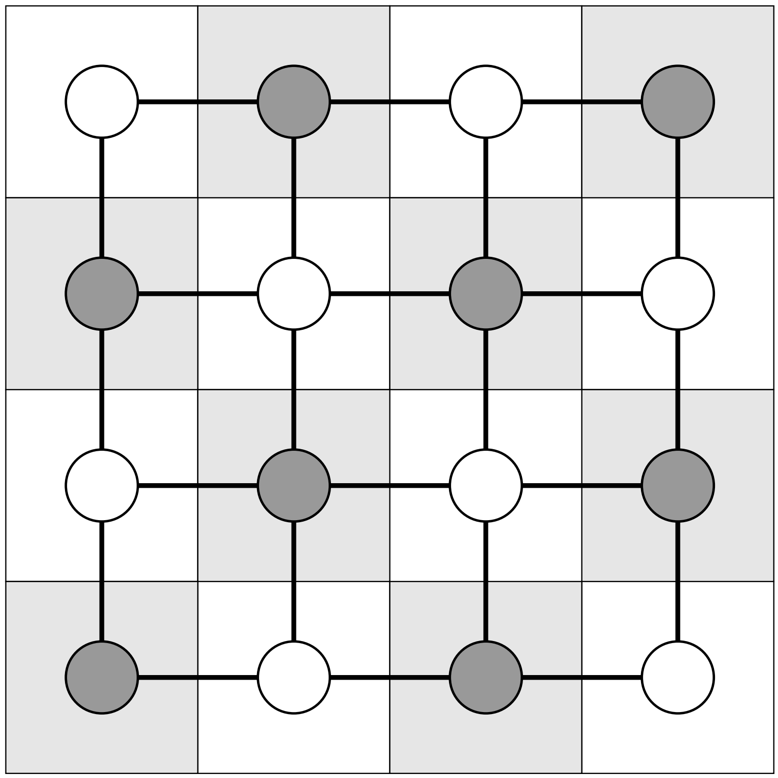
Example:



Solution: (a perfect matching)

The Dual Graph

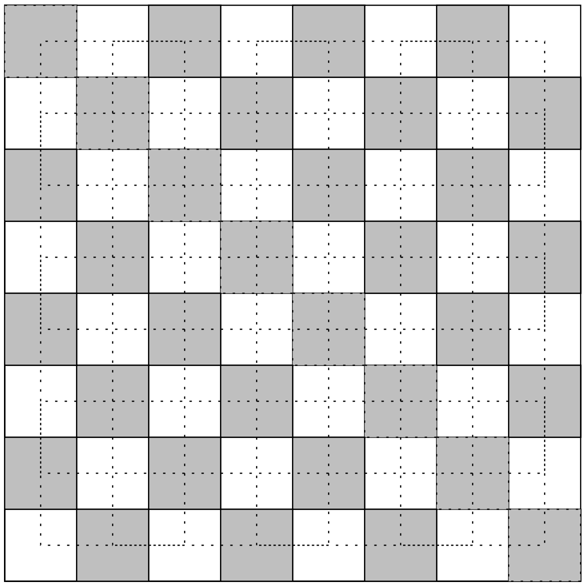
Given any region, we can create its dual graph.

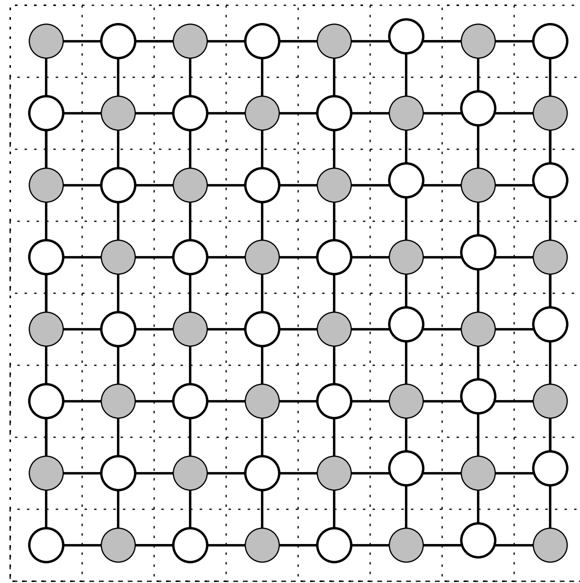


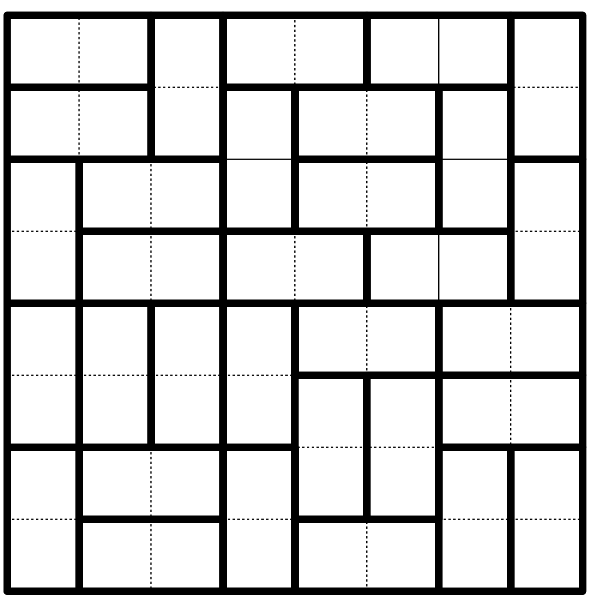
Place a vertex in every square; connect vertices whose squares are adjacent.

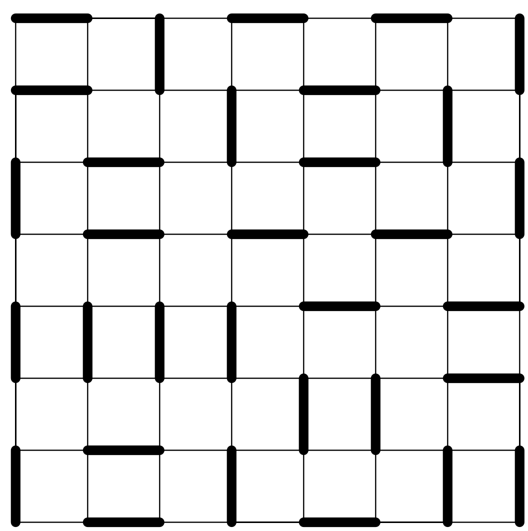
The dual graph of this region is bipartite.

Correspondence









The Determinant's Little Brother

The determinant of a matrix M:

$$\det M = \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) m_{1,\sigma(1)} m_{2,\sigma(2)} \cdots m_{k,\sigma(k)}.$$

The permanent of a matrix M:

$$\operatorname{perm} M = \sum_{\sigma \in S_k} m_{1,\sigma(1)} m_{2,\sigma(2)} \cdots m_{k,\sigma(k)}.$$

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$

No permanent calculus exists.

Example:

$$\operatorname{perm} \begin{pmatrix} 7 & 6 \\ 1 & 2 \end{pmatrix} = 7 \cdot 2 + 6 \cdot 1 = 20$$

Using Permanents

Entry $m_{ij} = \left\{ \begin{array}{ll} 1 & v_i b_j \in E(G) \\ 0 & v_i b_j \notin E(G) \end{array} \right\}$

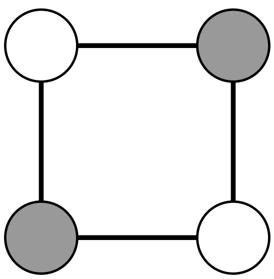
$$\begin{aligned}
\text{perm}\, M &= \text{Sum of terms of the form} \\
&\qquad m_{1,\sigma(1)} m_{2,\sigma(2)} \cdots m_{N,\sigma(N)} \\
&= \#\text{ of non-zero terms of the form} \\
&\qquad m_{1,\sigma(1)} m_{2,\sigma(2)} \cdots m_{N,\sigma(N)} \\
&= \#\text{ of choices of } N \text{ non-zero entries in } M \\
&= \#\text{ of choices of } N \text{ edges in the dual graph} \\
&= \#\text{ of perfect matchings in the dual graph}
\end{aligned}$$

A Kasteleyn–Percus Matrix

Convert the permanent to a determinant.

On a square lattice, the rule is easy to implement.

A toy example:



$$\rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

↓	↓	↓
2 perfect matchings	permanent $= 2$	determinant $= 2$

For a 4×4 Chessboard

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}$$

Comparison of Methods

	G–V	K–P
for AD_n	$n \times n$ entries need calculation	$n(n+1) \times n(n+1)$ entries are $0, \pm 1$ in predictable manner.

Proving a Kasteleyn Result

Theorem (H, 2005). Let G be the dual graph of a *nice region*. The Kasteleyn-Percus matrix A of G is alternating pseudo-centrosymmetric.

Theorem (H, 2005). Let A be alternating pseudo-centrosymmetric with entries in $\mathbb{Z}$. Then $\det A$ is a sum of two integral squares.

Open Problems

- Calculating the sequence of determinants explicitly. (Goal: closed form)

- Horizontal versus Vertical applications of Gessel–Viennot (Intriguing similarity)

- Learn more about combinatorial properties from matrix theory

Thanks!

I am: Christopher Hanusa

http://www.math.binghamton.edu/chanusa/

Additional reading:

Gessel–Viennot:
Lattice Paths and Determinants, by Martin Aigner

Kasteleyn–Percus:
Kasteleyn Cokernels, by Greg Kuperberg

Problems in Matching Theory:
Enumeration of Matchings: Problems and Progress, by James Propp

Orientations