Combinatorial interpretations in affine Coxeter groups

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What is a Coxeter group?

A **Coxeter group** is a group with

- Generators: $S = \{s_1, s_2, \ldots, s_n\}$
- Relations: $s_i^2 = 1, \quad (s_i s_j)^{m_{i,j}} = 1$ where $m_{i,j} \geq 2$ or $= \infty$
  - $m_{i,j} = 2$: $(s_i s_j)(s_i s_j) = 1 \quad \longrightarrow \quad s_i s_j = s_j s_i$ (they commute)
  - $m_{i,j} = 3$: $(s_i s_j)(s_i s_j)(s_i s_j) = 1 \rightarrow s_i s_j s_i = s_j s_i s_j$ (braid relation)
  - $m_{i,j} = \infty$: s_i and s_j are not related.

Why Coxeter groups?

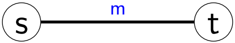
- They're awesome.
- Discrete Geometry: Symmetries of regular polyhedra.
- Algebra: Symmetric group generalizations. (Kac-Moody, Hecke)
- Geometry: Classification of Lie groups and Lie algebras

Examples of Coxeter groups

A shorthand notation is the **Coxeter graph**:

- Vertices: One for every generator i
- Edges: Create an edge between i and j when $m_{i,j} \geq 3$
  Label edges with $m_{i,j}$ when ≥ 4.

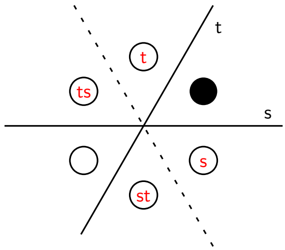
Dihedral group



- Generators: s, t.
- Relation: $(st)^m = 1$.

Symmetry group of regular m-gon.

When $m = 3$:



Examples of Coxeter groups

(Finite) n-permutations S_n

An n-permutation is a permutation of $\{1, 2, \ldots, n\}$, (e.g. 2 1 4 5 3 6).

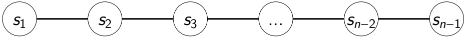
Every n-permutation is a product of *adjacent transpositions*.

- $s_i : (i) \leftrightarrow (i+1)$. (e.g. $s_4 = 1\,2\,3\,5\,4\,6$).

Example. Write 2 1 4 5 3 6 as $s_3 s_4 s_1$.

This is a Coxeter group:

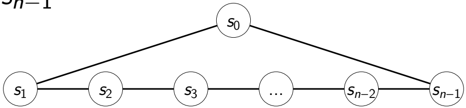
- Generators: $s_1, \ldots, s_{n-1}$
- $s_i s_j = s_j s_i$ when $|i-j| \geq 2$ (commutation relation)
- $s_i s_j s_i = s_j s_i s_j$ when $|i-j| = 1$ (braid relation)



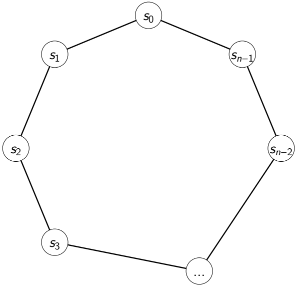
Examples of Coxeter groups

Affine n-Permutations $\widetilde{S_n}$

- Generators: $s_0, s_1, \ldots, s_{n-1}$
- Relations:



- s_0 has a braid relation with s_1 and s_{n-1}
- How does this impact 1-line notation?
  - Perhaps interchanges 1 and n?
  - Not quite! (Would add a relation)
- Better to view graph as:
  - Every generator is the same.



Examples of Coxeter groups

Affine n-Permutations $\widetilde{S}_n$ (G. Lusztig 1983, H. Eriksson, 1994)

Write an element $\widetilde{w} \in \widetilde{S}_n$ in 1-line notation as a permutation of $\mathbb{Z}$.

Generators transpose **infinitely many** pairs of entries:

s_i : **(i)** $\leftrightarrow$ **(i+1)** $\ldots (n+i) \leftrightarrow (n+i+1) \ldots (-n+i) \leftrightarrow (-n+i+1) \ldots$

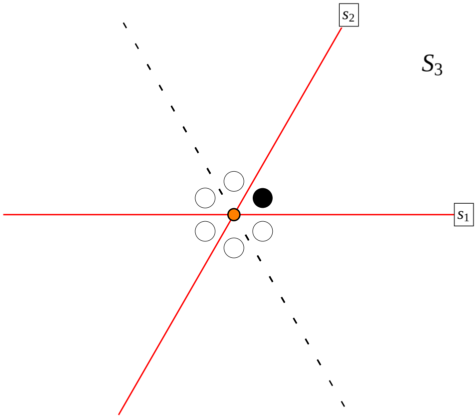
In $\widetilde{S}_4$,	$\cdots w(-4)$	$w(-3)$	$w(-2)$	$w(-1)$	$w(0)$	$w(1)$	$w(2)$	$w(3)$	$w(4)$	$w(5)$	$w(6)$	$w(7)$	$w(8)$	$w(9)\cdots$
s_1	… -4	-2	-3	-1	0	2	1	3	4	6	5	7	8	10 …
s_0	… -3	-4	-2	-1	1	0	2	3	5	4	6	7	9	8 …
$s_1 s_0$	… -2	-4	-3	-1	2	0	1	3	6	4	5	7	10	8 …

Symmetry: Can think of as integers wrapped around a cylinder.

$\widetilde{w}$ is defined by the window $[\widetilde{w}(1), \widetilde{w}(2), \ldots, \widetilde{w}(n)]$. $s_1 s_0 = [0, 1, 3, 6]$

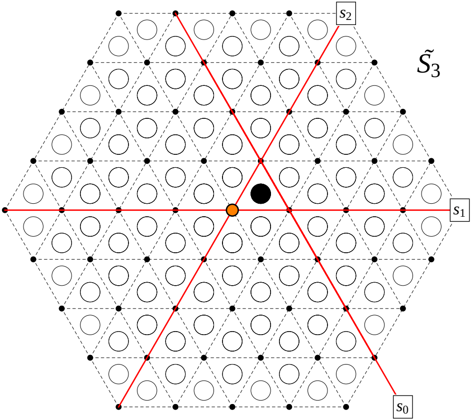
Examples of Coxeter groups

Affine n-Permutations $\widetilde{S_n}$



Examples of Coxeter groups

Affine *n*-Permutations $\widetilde{S_n}$ — elements correspond to alcoves.



Properties of Coxeter groups

For a elements w in a Coxeter group W,

- w may have multiple expressions.
  - Transfer between them using relations.

Example. In S_4, $w = s_1 s_2 s_3 s_1 = s_1 s_2 s_1 s_3 = s_2 s_1 s_2 s_3 = s_2 s_1 s_2 s_3 s_1 s_1$

- w has a shortest expression (this length: **Coxeter length**)

For a Coxeter group $\widetilde{W}$,

- An induced subgraph of $\widetilde{W}$'s Coxeter graph is a subgroup W
- Every element $\widetilde{w} \in \widetilde{W}$ can be written $\widetilde{w} = w^0 w$, where $w^0 \in \widetilde{W}/W$ is a coset representative and $w \in W$.

S_n as a subgroup of $\widetilde{S}_n$

Key concept: View S_n as a subgroup of $\widetilde{S_n}$.

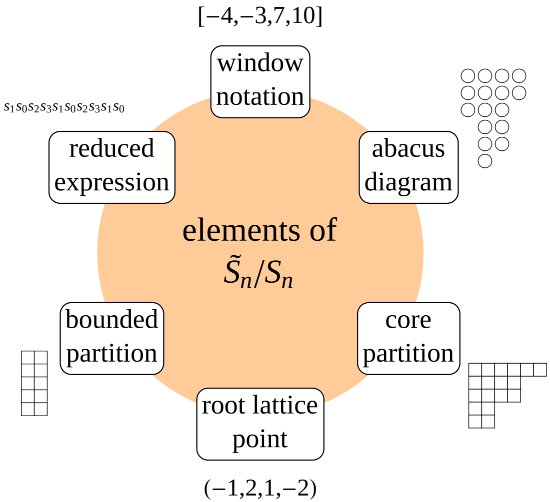
- Write $\widetilde{w} = w^0 w$, where $w^0 \in \widetilde{S_n}/S_n$ and $w \in S_n$.
  - w^0 **determines the entries;** w **determines their order.**

Example. For $\widetilde{w} = [-11, 20, -3, 4, 11, 0] \in \widetilde{S}_6$,

$$w^0 = [-11, -3, 0, 4, 11, 20] \text{ and } w = [1, 3, 6, 4, 5, 2].$$

Many interpretations of these *minimal length coset representatives*.

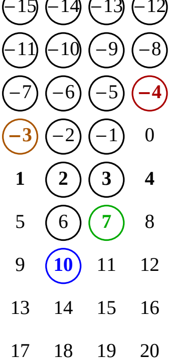
Combinatorial interpretations of $\widetilde{S}_n/S_n$



An abacus model for $\widetilde{S}_n/S_n$

(James and Kerber, 1981) Given $w^0 = [w_1, \dots, w_n] \in \widetilde{S}_n/S_n$,

- Place integers in n *runners*.
- Circled: *beads*. Empty: *gaps*
- Bijection: Given w^0, create an abacus where each runner has a lowest bead at w_i.



Example: $[-4, -3, 7, 10]$

These abaci are flush and balanced.

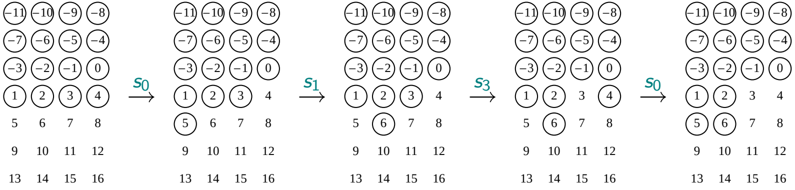
The generators act nicely on the abacus.

Action of generators on the abacus

- s_i acts by interchanging runners i and $i+1$.
- s_0 acts by interchanging runners 1 and n, with level shifts.

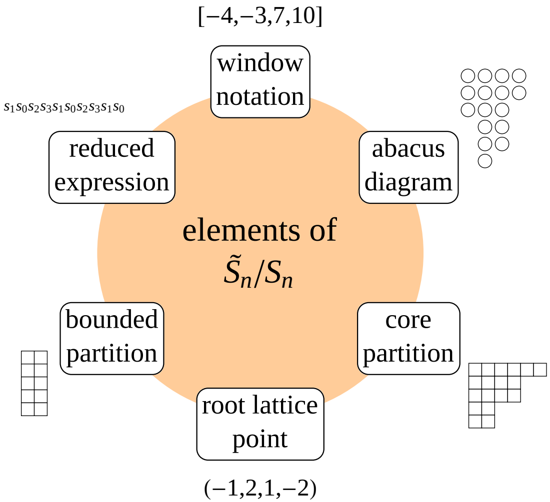
Example: Consider $[-4, -3, 7, 10] = s_1 s_0 s_2 s_1 s_3 s_2 s_0 s_3 s_1 s_0$.

Start with id$= [1, 2, 3, 4]$ and apply the generators one by one:



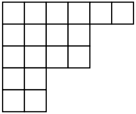
$[1, 2, 3, 4]$ $\quad [0, 2, 3, 5]$ $\quad [0, 1, 3, 6]$ $\quad [-1, 1, 4, 6]$ $\quad [-1, 0, 5, 6]$

Combinatorial interpretations of $\widetilde{S}_n/S_n$



Integer partitions and n-core partitions

For an integer partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ drawn as a Ferrers diagram,



The *hook length* of a box is $\#$ boxes below and to the right.

10	9	6	5	2	1
7	6	3	2		
6	5	2	1		
3	2				
2	1				

An *n-core* is a partition with no boxes of hook length dividing n.

Example. λ is a 4-core, 8-core, 11-core, 12-core, etc.
λ is NOT a 1-, 2-, 3-, 5-, 6-, 7-, 9-, or 10-core.

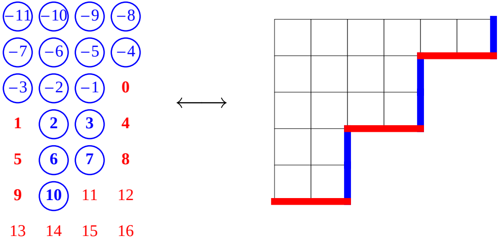
Core partitions for $\widetilde{S}_n/S_n$

Elements of $\widetilde{S}_n/S_n$ are in bijection with n-cores.

Bijection: $\{\text{abaci}\} \longleftrightarrow \{n\text{-cores}\}$

Rule: Read the boundary steps of λ from the abacus:

- A bead $\leftrightarrow$ vertical step
- A gap $\leftrightarrow$ horizontal step

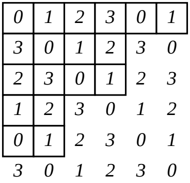


Fact: Abacus flush with n-runners $\leftrightarrow$ partition is n-core.

Action of generators on the core partition

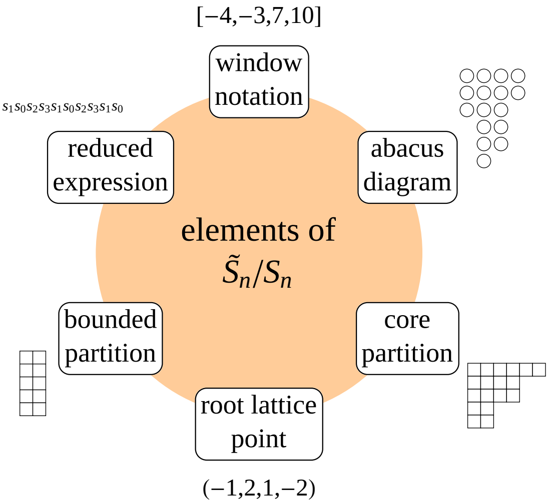
- Label the boxes of λ with residues.
- s_i acts by adding or removing boxes with residue i.

Example: Let's see the *deconstruction* of $s_1 s_0 s_2 s_1 s_3 s_2 s_0 s_3 s_1 s_0$:



Applying generator s_1
removes all removable 1-boxes.

Combinatorial interpretations of $\widetilde{S}_n/S_n$



Bounded partitions for $\widetilde{S}_n/S_n$

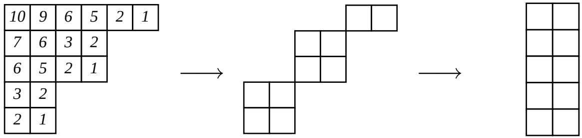
A partition $\beta = (\beta_1, \ldots, \beta_k)$ is *b-bounded* if $\beta_i \leq b$ for all i.

Elements of $\widetilde{S}_n/S_n$ are in bijection with $(n-1)$-bounded partitions.

Bijection: (Lapointe, Morse, 2005)

$$\{n\text{-cores } \lambda\} \leftrightarrow \{(n-1)\text{-bounded partitions } \beta\}$$

- Remove all boxes of λ with hook length $\geq n$
- Left-justify remaining boxes.



$\lambda = (6, 4, 4, 2, 2)$ $\beta = (2, 2, 2, 2, 2)$

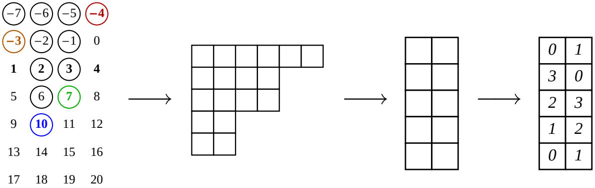
Canonical reduced expression for $\widetilde{S}_n/S_n$

Given the bounded partition, read off the reduced expression:

Method: (Berg, Jones, Vazirani, 2009)

- Fill β with residues i
- Tally s_i reading right-to-left in rows from bottom-to-top

Example. $[-4, -3, 7, 10] = s_1 s_0 s_2 s_1 s_3 s_2 s_0 s_3 s_1 s_0$.



- The Coxeter length of w is the number of boxes in β.

Fully commutative elements

Definition. An element in a Coxeter group is **fully commutative** if it has only one reduced expression (up to commutation relations).

NO BRAIDS ALLOWED!

Example. In S_4, $s_1 s_2 s_3 s_1$ is **not fully commutative** because

$$s_1 s_2 s_3 s_1 \overset{\text{OK}}{=} s_1 s_2 s_1 s_3 \overset{\text{BAD}}{=} s_2 s_1 s_2 s_3$$

Question: What is $s_1 s_2 s_1$ in 1-line notation?

Answer: 3 2 1 4 5 6 . . .

Enumerating fully commutative elements

Question: **How many** fully commutative elements are there in S_n?

Answer: Catalan many!

S_1: **1.** id

S_2: **2.** id, s_1

S_3: **5.** id, s_1, s_2, s_1s_2, s_2s_1

S_4: **14.** id, s_1, s_2, s_3, s_1s_2, s_2s_1, s_2s_3, s_3s_2, s_1s_3, $s_1s_2s_3$, $s_1s_3s_2$, $s_2s_1s_3$, $s_3s_2s_1$, $s_2s_1s_3s_2$

Key idea: (Billey, Jockusch, Stanley, 1993)

$$w \text{ is fully commutative} \iff w \text{ is 321-avoiding.}$$

(Knuth, 1973) These are counted by the Catalan numbers.

Enumerating fully commutative elements

Question: **How many** fully commutative elements are there in $\widetilde{S_n}$?

Answer: Infinitely many! (Even in $\widetilde{S}_3$.)

$$\mathrm{id}, s_1, s_1 s_2, s_1 s_2 s_0, s_1 s_2 s_0 s_1, s_1 s_2 s_0 s_1 s_2, \ldots$$

Multiplying the generators cyclically does not introduce braids.

This is not the right question.

Enumerating fully commutative elements

Question: **How many** fully commutative elements are there in $\widetilde{S}_n$, with Coxeter length ℓ?

$$\text{In } \widetilde{S}_3: \quad \text{id}, \quad \begin{matrix} s_0 \\ s_1 \\ s_2 \end{matrix}, \quad \begin{matrix} s_0 s_1 & s_0 s_2 \\ s_1 s_0 & s_1 s_2 \\ s_2 s_0 & s_2 s_1 \end{matrix}, \quad \begin{matrix} s_0 s_1 s_2 & s_0 s_2 s_1 \\ s_1 s_0 s_2 & s_1 s_2 s_0 \\ s_2 s_0 s_1 & s_2 s_1 s_0 \end{matrix}, \ldots$$

Question: Determine the coefficient of q^ℓ in the generating function

$$f_n(q) = \sum_{\widetilde{w} \in \widetilde{S}_n^{FC}} q^{\ell(w)}.$$

$$f_3(q) = 1q^0 + 3q^1 + 6q^2 + 6q^3 + \ldots$$

Answer: Consult your friendly computer algebra program.

DdddaaaaAAAAaaaaTTaaaaAA

Brant calls up and says: "Hey Chris, look at this data!"

$$f_3(q) = 1 + 3q + \mathbf{6}q^2 + \mathbf{6}q^3 + \mathbf{6}q^4 + \mathbf{6}q^5 + \cdots$$
$$f_4(q) = 1 + 4q + 10q^2 + \mathbf{16}q^3 + \mathbf{18}q^4 + \mathbf{16}q^5 + \mathbf{18}q^6 + \cdots$$
$$f_5(q) = 1 + 5q + 15q^2 + 30q^3 + 45q^4 + \mathbf{50}q^5 + \mathbf{50}q^6 + \mathbf{50}q^7 + \mathbf{50}q^8 + \cdots$$
$$f_6(q) = 1 + 6q + 21q^2 + 50q^3 + 90q^4 + 126q^5 + 146q^6 + \mathbf{150}q^7 + \mathbf{156}q^8 + \mathbf{152}q^9 + \mathbf{156}q^{10} + \mathbf{150}q^{11} + \mathbf{158}q^{12} + \mathbf{150}q^{13} + \mathbf{156}q^{14} + \mathbf{152}q^{15} + \mathbf{156}q^{16} + \mathbf{150}q^{17} + \mathbf{158}q^{18} + \cdots$$
$$f_7(q) = 1 + 7q + 28q^2 + 77q^3 + 161q^4 + 266q^5 + 364q^6 + 427q^7 + 462q^8 + 483q^9 + \mathbf{490}q^{10} + \mathbf{490}q^{11} + \mathbf{490}q^{12} + \mathbf{490}q^{13} + \cdots$$

Notice:

- The coefficients eventually repeat.

Goals: ★ Find a formula for the generating function $f_n(q)$.
★ Understand this periodicity.

Pattern Avoidance Characterization

Key idea: (**Green**, 2002)

$\widetilde{w}$ is fully commutative $\iff$ $\widetilde{w}$ is 321-avoiding.

Example. $[-4, -1, 1, 14]$ is **NOT** fully commutative because:

	$\cdots w(-4)$	$w(-3)$ $w(-2)$ $w(-1)$ $w(0)$	$w(1)$ $w(2)$ $w(3)$ $w(4)$	$w(5)$ $w(6)$ $w(7)$ $w(8)$	$w(9) \cdots$
$\widetilde{w}$	$\ldots$ 6	-8 -5 -3 **10**	-4 -1 **1** 14	**0** 3 5 18	4 $\ldots$

Game plan

Goal: Enumerate 321-avoiding affine permutations $\widetilde{w}$.

- Write $\widetilde{w} = w^0 w$, where $w^0 \in \widetilde{S_n}/S_n$ and $w \in S_n$.
  - w^0 **determines the entries;** w **determines their order.**

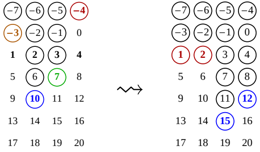
Example. For $\widetilde{w} = [-11, 20, -3, 4, 11, 0] \in \widetilde{S}_6$,

$$w^0 = [-11, -3, 0, 4, 11, 20] \text{ and } w = [1, 3, 6, 4, 5, 2].$$

- Determine which w^0 are 321-avoiding.
- Determine the finite w such that $w^0 w$ is still 321-avoiding

Normalized abacus and 321-avoiding criterion for $\widetilde{S}_n/S_n$

We use a *normalized* abacus diagram; shifts all beads so that the first gap is in position $n+1$; this map is invertible.



Theorem. (H–J '09) Given a normalized abacus for $w^0 \in \widetilde{S}_n/S_n$, where the last bead occurs in position i,

w^0 is fully commutative $\iff$ lowest beads in runners only occur in $\{1,\ldots,n\} \cup \{i-n+1,\ldots,i\}$

Idea: Lowest beads in runners $\leftrightarrow$ entries in base window.

w(-n+1)	w(-n+2)	...	w(-1)	w(0)	w(1)	w(2)	...	w(n-1)	w(n)	w(n+1)	w(n+2)	...	w(2n-1)	w(2n)
lo	lo	...	hi	hi	lo	lo	...	hi	hi	lo	lo	...	hi	hi
lo	lo	med	hi	**hi**	lo	lo	**med**	hi	hi	**lo**	lo	med	hi	hi

Long versus short elements

Partition $\widetilde{S_n}$ into long and short elements:

Short elements

Lowest bead in position $i \leq 2n$

Finitely many

Hard to count

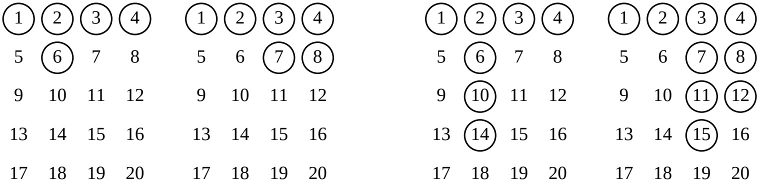
Long elements

Lowest bead in position $i > 2n$

Come in infinite families

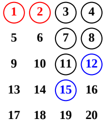
Easy to count

Explain the periodicity



Enumerating long elements

For long elements $\widetilde{w} \in \widetilde{S_n}$, the base window for w^0 is $[a, a, \ldots, a, b, b, \ldots, b]$ where $1 \leq a \leq n$, and $n + 2 \leq b$.



Question: Which permutations $w \in S_n$ can be multiplied into a w^0?

- We can not invert any pairs of a's, nor any pairs of b's. (Would create a 321-pattern with an adjacent window)
- Only possible to *intersperse* the a's and the b's.

How many ways to intersperse (k) a's and $(n-k)$ b's? $\binom{n}{k}$

BUT: We must also keep track of the *length* of these permutations. This is counted by the q-binomial coefficient: $\begin{bmatrix} n \\ k \end{bmatrix}_q$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q)_n}{(q)_k (q)_{n-k}}, \text{ where } q_n = (1-q)(1-q^2)\cdots(1-q^n)$$

Enumerating long elements

After we:

- Enumerate by length all possible w^0 with (k) a's and $(n-k)$ b's.
- Combine the Coxeter lengths by $\ell(\widetilde{w}) = \ell(w^0) + \ell(w)$.

Then we get:

Theorem. (H–J '09) For a fixed $n \geq 0$, the generating function by length for *long* fully commutative elements $\widetilde{w} \in \widetilde{S}_n^{FC}$ is

$$\sum q^{\ell(\widetilde{w})} = \frac{q^n}{1-q^n} \sum_{k=1}^{n-1} {\begin{bmatrix} n \\ k \end{bmatrix}_q}^2 .$$

Periodicity of fully commutative elements in $\widetilde{S}_n$

Corollary. (H–J '09) The coefficients of $f_n(q)$ are eventually periodic with period dividing n.

When n is prime, the period is 1: $a_i = \frac{1}{n}\left(\binom{2n}{n} - 2\right)$.

Proof. For i sufficiently large, all elements of length i are long. Our generating function is simply some polynomial over $(1 - q^n)$:

$$\frac{q^n}{1-q^n}\sum_{k=1}^{n-1}\begin{bmatrix} n \\ k \end{bmatrix}_q^2 = \frac{P(q)}{1-q^n} = P(q)(1 + q^n + q^{2n} + \cdots)$$

When n is prime, an extra factor of $(1 + q + \cdots + q^{n-1})$ cancels;

$$\frac{1}{1-q}\left[\frac{q^n}{1+q+\cdots+q^{n-1}}\sum_{k=1}^{n-1}\begin{bmatrix} n \\ k \end{bmatrix}_q^2\right]$$

As suggested by a referee, we know that $a_i = P(1) = \frac{1}{n}\sum_{k=1}^{n-1}\binom{n}{k}^2$.

Short elements are hard

For short elements $\widetilde{w} \in \widetilde{S_n}$, the base window for w^0 is $[a, \ldots, a, b, \ldots, b, c, \ldots, c]$, and there is more interaction:

(1)	(2)	(3)	(4)
5	(6)	7	8
9	10	11	12

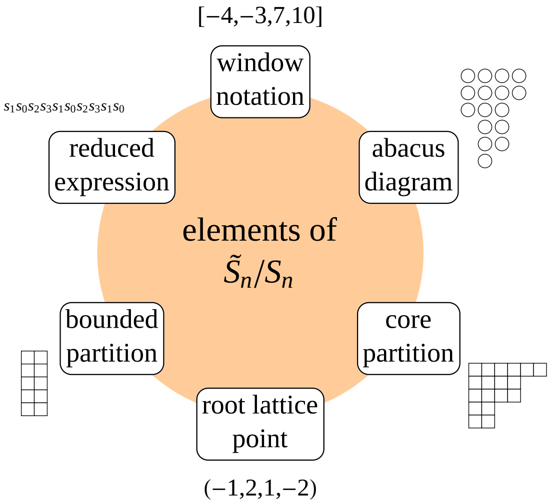
No a can invert with an a or b. No c can invert with a b or c.

- Count $\widetilde{w}$ where some a intertwines with some c.
- Count $\widetilde{w}$ w/o intertwining and 0 descents in the b's.
- Count $\widetilde{w}$ w/o intertwining and 1 descent in the b's.
  - Not so hard to determine the acceptable finite permutations w.
  - Such as $\sum_{M\geq 0} x^{L+M+R} \sum_{\mu=1}^{M-1} \left(\begin{bmatrix} M \\ \mu \end{bmatrix}_q - 1 \right) \begin{bmatrix} L+\mu \\ \mu \end{bmatrix}_q \begin{bmatrix} R+M-\mu \\ M-\mu \end{bmatrix}_q$
- Count $\widetilde{w}$ w/o intertwining and 2 descents in the b's.
- Count $\widetilde{w}$ which are finite permutations. (Barcucci et al.)
  - Solve functional recurrences (Bousquet-Mélou)
  - Such as $D(x, q, z, s) =$
    $N(x, q, z, s) + \frac{xqs}{1-qs}\big(D(x, q, z, 1) - D(x, q, z, qs)\big) + xsD(x, q, z, s)$

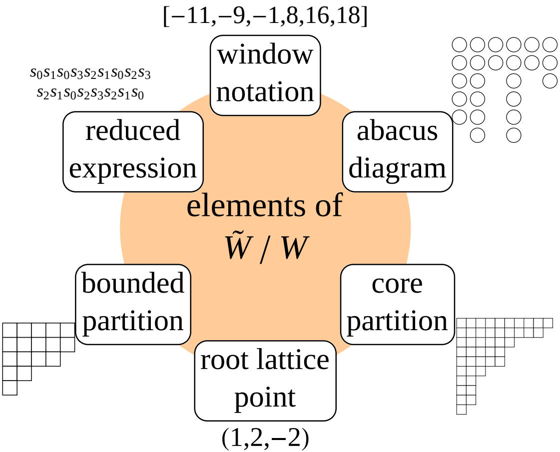
Future Work

- Extend to $\widetilde{B}_n$, $\widetilde{C}_n$, and $\widetilde{D}_n$
  - Develop combinatorial interpretations ✓
  - 321-avoiding characterization?
- Heap interpretation of fully commutative elements
  - Can use Viennot's heaps of pieces theory
  - Better bound on periodicity
- More combinatorial interpretations for $\widetilde{W}/W$
  - What do you know?

Combinatorial interpretations of $\widetilde{S}_n/S_n$



Combinatorial interpretations of $\widetilde{C}/C$, $\widetilde{B}/B$, $\widetilde{B}/D$, $\widetilde{D}/D$



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Thank you!

Slides available: `people.qc.cuny.edu/chanusa` > `Talks`

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