A quasi-polynomial q-Queens result and related Kronecker products of matrices

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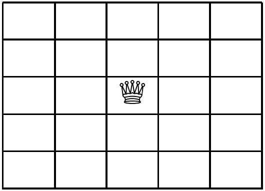
The n-Queens Problem

Motivating question:
Can you place n nonattacking queens on an $n \times n$ chessboard?



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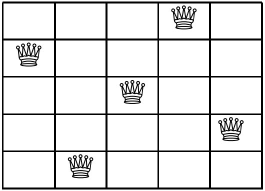


Q: In how many ways can you place n nonattacking queens?

n	1	2	3	4	5	6	7	8	9	10
#	1	0	0	2	10	4	40	92	352	724

The q-Queens Problem and generalizations

Let's generalize.

- Fix the number of queens. (q)
- Let the size of the board vary. ($n \times n$)

Question: Determine the number of ways in which you can place q nonattacking queens on an $n \times n$ chessboard as a function of n.

Question: Why stop there?

The q-Queens Problem and generalizations

A problem will have three elements:

- A piece. (A set of basic moves.)
- A board. (A convex polygon and its dilations.)
- A number. (A number of pieces to arrange.)

A **piece** P moves from $z = (x, y)$ to $z + \alpha m_r$ for $m_r \in \mathbf{M}$, $\alpha \in \mathbb{Z}$

Two pieces in z_i and z_j are attacking if $z_i - z_j = \alpha m_r$.

Examples:

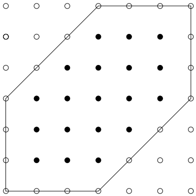
- ♕ Queens: $\mathbf{M} = \{(1,0),(1,1),(0,1),(1,-1)\}$
- ♗ Bishops: $\mathbf{M} = \{(1,1),(1,-1)\}$
- ♘ Nightrider: $\mathbf{M} = \{(2,1),(1,2),(2,-1),(1,-2)\}$

The q-Queens Problem and generalizations

A problem will have three elements:

- A piece. (A set of basic moves.)
- A board. (A convex polygon and its dilations.)
- A number. (A number of pieces to arrange.)

A **board** is the set of integral points on the interior of an integral multiple of a rational convex polygon $\mathcal{B} \subset \mathbb{R}^2$



The q-Queens Problem and generalizations

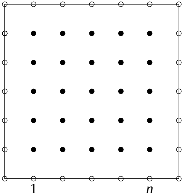
Question: Given a piece P, a polygon $\mathcal{B}$, and a number q,

Determine the number of ways in which you can place q nonattacking P pieces on the board $t\mathcal{B}^\circ$ as a function of t.

In the original q-Queens Problem,

- $P =$ ♕
- $\mathcal{B} = [0,1]^2$
- $q = q$

The $n \times n$ case corresponds to $t = (n+1)$.



The q-Queens Problem and generalizations

Theorem: (Chaiken, Zaslavsky, 2005)
Given P, $\mathcal{B}$, and q, the number of placements of q nonattacking P pieces inside $t\mathcal{B}$ is a quasipolynomial function of t.

A *quasipolynomial* is a function $f(t)$ on $t \in \mathbb{Z}_+$ such that

$$f(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_0,$$

where each c_i is periodic.

Example: $f(t) = \begin{cases} t^2 + 3t + 2 & \text{for even } t \\ t^2 - 2t + 1 & \text{for odd } t \end{cases}$

q-Queens proof sketch

Very briefly:

The rules of nonattack correspond to forbidden hyperplanes in $\mathbb{R}^{2q}$.

Inside-out polytope theory gives a quasipolynomial function of t.

q-Queens proof sketch

Less briefly:

- Goal: Count allowed unordered configuration of pieces.
- Instead, count allowed *ordered* configurations of $z_i = (x_i, y_i)$.

- A configuration is a point $(x_1, y_1, \ldots, x_q, y_q) \in \mathbb{Z}^{2q} \cap t\mathcal{B}^q$
- Two pieces are attacking when $(z_j - z_i) \cdot m_r^\perp = 0$.
- There are $\binom{q}{2}N$ of these forbidden hyperplanes in $\mathbb{R}^{2q}$

- Count lattice points in $t\mathcal{B}^q$ avoiding $\mathcal{H}$.
- This is a direct application of inside-out polytope theory
  Counted by a quasipolynomial with certain properties.

Inside-out polytopes

(Beck, Zaslavsky, 2006) An *inside-out polytope* $(\mathcal{P}, \mathcal{H})$

- Builds upon ideas of Ehrhart theory.
- $\mathcal{P}$ is a convex polytope
  - Vertices of $\mathcal{P}$ have rational coordinates
- $\mathcal{H}$ is an arrangement of hyperplanes dissecting $\mathcal{P}$.
  - The $\mathcal{H}$ have rational equations.
  - The $\mathcal{H}$ are homogeneous.
- Counts (t^{-1})-fractional points inside $\mathcal{P}$.

integral points inside $t\mathcal{B}^q$	$\longleftrightarrow$	t^{-1}-fractional points inside $\mathcal{B}^q$

Inside-out polytopes

Conclusion: The number of lattice points inside $\mathcal{P}$ avoiding $\mathcal{H}$ is a quasipolynomial function of t with

- degree: $\dim(\mathcal{P})$.
- leading coefficient: volume of $\mathcal{P}$.

Therefore: The number nonattacking configurations of q pieces P inside $t\mathcal{B}$ is a quasipolynomial function of t with

- degree: $\dim(\mathcal{B}^q) = 2q$.
- leading coefficient: $|\mathcal{B}|^q/q!$. $\longleftarrow$ Now unordered!

But what does this mean?

So, we have a solution to q-Queens and n-Queens?

- No. The theorem only proves existence.

We must determine the periodic coefficients c_i.

Game plan:

- Determine the period of the coefficients.
- Compute initial data to determine the formula.

Rooks and bishops

Notation: Write $u_P(q;n)$ for the number of (unlabeled) nonattacking configurations of q pieces P on an $n \times n$ board.

Translation: $\mathcal{B} = [0,1]^2$ and $t = n+1$, implying:

- degree of $u_P(q;n)$ is $2q$.
- leading coefficient of $u_P(q;n)$ is $1/q!$.

For a fixed q, we expect a formula of the form:

$$u_P(q;n) = \frac{1}{q!}n^{2q} + c_{2q-1}n^{2q-1} + \cdots + c_1 n + c_0$$

Classic result for rooks R:

♖ $u_R(q;n) = q!\binom{n}{q}^2$.

Rooks and bishops

♗ For bishops B:

- We will calculate that the period divides 2^{q-1}. (stay tuned!)

- One ♗: $u_B(1;n) = n^2$.
- Two ♗: quasipolynomial of degree 4, period 1 or 2.
- $u_B(2;n) = \frac{1}{2}n^4 + c_3 n^3 + c_2 n^2 + c_1 n + c_0$.

- Initial data for $u_B(2;n)$:

n	1	2	3	4	5	6	7	8
$u_B(2;n)$	0	4	26	92	240	520	994	1736

$$u_B(2;n) = \frac{1}{2}n^4 - \frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{1}{3}n.$$

Rooks and bishops

♗ (CHZ 20??) For bishops B, the period divides 2.

$$u_B(1;n) = n^2.$$

$$u_B(2;n) = \frac{n^4}{2} - \frac{2n^3}{3} + \frac{n^2}{2} - \frac{n}{3}$$

$$u_B(3;n) = \left\{\frac{n^6}{6} - \frac{2n^5}{3} + \frac{5n^4}{4} - \frac{5n^3}{3} + \frac{4n^2}{3} - \frac{2n}{3} + \frac{1}{8}\right\} - (-1)^n\frac{1}{8}.$$

$$u_B(4;n) = \left\{\frac{n^8}{24} - \frac{n^7}{3} + \frac{11n^6}{9} - \frac{29n^5}{10} + \frac{355n^4}{72} - \frac{35n^3}{6} + \frac{337n^2}{72} - \frac{73n}{30} + \frac{1}{2}\right\} - (-1)^n\left\{\frac{n^2}{8} - \frac{n}{2} + \frac{1}{2}\right\}.$$

$$u_B(5;n) = \left\{\frac{n^{10}}{120} - \frac{n^9}{9} + \frac{49n^8}{72} - \frac{118n^7}{45} + \frac{523n^6}{72} - \frac{2731n^5}{180} + \frac{3413n^4}{144} - \frac{4853n^3}{180} + \frac{2599n^2}{120} - \frac{1321n}{120} + \frac{9}{4}\right\} - (-1)^n\left\{\frac{n^4}{16} - \frac{7n^3}{12} + \frac{17n^2}{8} - \frac{85n}{24} + \frac{9}{4}\right\}.$$

Finding the quasipolynomial period

- The period of the quasipolynomial depends on the vertices of the inside-out polytope.
- If a vertex has denominator $d \rightsquigarrow$ the period depends on d.
- Expect: Period divides the lcm over all d_v.

Question. How to find the vertices?

Answer. Linear algebra!

Use a matrix to determine intersections of

- The forbidden hyperplanes (for each move $m_r^\perp = (m_{r1}, m_{r2})$)
  - Equations: $m_r^\perp \cdot ((x_j, y_j) - (x_i, y_i)) = 0$
- The faces of the polytope (defined by $a_j x + b_j y \leq \beta_j$)
  - Equations: $(a_j, b_j) \cdot (x_i, y_i) \leq \beta_j$

Finding the quasipolynomial period

$$
\left(\begin{array}{ccccccc}
M & -M & 0 & 0 & \cdots & 0 & 0 \\
M & 0 & -M & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
M & 0 & 0 & 0 & \cdots & 0 & -M \\
0 & M & -M & 0 & \cdots & 0 & 0 \\
0 & M & 0 & -M & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & M & 0 & 0 & \cdots & 0 & -M \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & M & -M \\
\hline
B & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & B & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & B
\end{array}\right)
\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_q \end{pmatrix}
=
\left(\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \hline \beta \\ \beta \\ \vdots \\ \beta \end{array}\right)
$$

where M is the **moves matrix**

$$M := \begin{pmatrix} m_1^{\perp} \\ m_2^{\perp} \\ \vdots \\ m_{|\mathbf{M}|}^{\perp} \end{pmatrix}$$

and

$$B := \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_K & b_K \end{pmatrix}.$$

- Cramer's Rule $\rightsquigarrow$ vertex denominator divides a subdet. of A.
- Period of quasipoly. divides **lcm of all such subdet's**, $\operatorname{lcmd}(A)$.
- A square board simplifies. The structure of A' is predictable.

The structure of A'

$$A' = \begin{pmatrix} M & -M & 0 & 0 & \cdots & 0 & 0 \\ M & 0 & -M & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ M & 0 & 0 & 0 & \cdots & 0 & -M \\ 0 & M & -M & 0 & \cdots & 0 & 0 \\ 0 & M & 0 & -M & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & M & 0 & 0 & \cdots & 0 & -M \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & M & -M \end{pmatrix}$$

This reminds us of the incidence matrix for the complete graph K_q:

$$D(K_q) = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & \cdots & 0 \\ 0 & -1 & \cdots & 0 & -1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 \\ 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & -1 & \cdots & -1 \end{pmatrix}$$

The structure of F^T

For matrices $A = (a_{ij})_{m\times k}$ and $B = (b_{ij})_{n\times l}$, the *Kronecker product* $A \otimes B$ is defined to be the $mn \times kl$ block matrix

$$\left[\begin{array}{c:c:c} a_{11}B & \cdots & a_{1k}B \\ \hdashline \vdots & \ddots & \vdots \\ \hdashline a_{m1}B & \cdots & a_{mk}B \end{array}\right].$$

We have $(A')^T = D(K_q) \otimes M^T$

- M is the $m \times 2$ moves matrix
- $D(K_q)$ is the incidence matrix for the complete graph K_q.

About Kronecker Products

About Kronecker products:

- $A \otimes B$ and $B \otimes A$ only differ by row and column switchings.
- For $A_{m\times m}$ and $B_{n\times n}$, $\det(A \otimes B) = \det(A)^n \det(B)^m$.

- Calculating $\operatorname{lcmd}(A \otimes B)$ appears difficult for generic A, B.
- We aim to simplify $\operatorname{lcmd}\left(M^T \otimes D(K_q)\right)$.
- Funny story.

lcmd result

Theorem (Hanusa, Zaslavsky, 2008) Given $M_{m\times 2}$ and $q \geq 1$,

$$\text{lcmd}\left(M \otimes D(K_q)\right) = \text{lcm}\left((\text{lcmd}\, M)^{q-1}, \underset{\mathcal{K}}{\text{LCM}}\left(\prod_{(I,J)\in\mathcal{K}} \det M^{I,J}\right)\right),$$

The LCM is over disjoint multisubsets I and J of $[m]$ of size $\lfloor q/2 \rfloor$...

- lcmd $\left(M \otimes D(K_q)\right)$ is simply an lcm over entries of M.

lcmd result

$$\operatorname{lcmd}(M \otimes D(K_q)) = \operatorname{lcm}\left((\operatorname{lcmd} M)^{q-1}, \underset{\mathcal{K}}{\operatorname{LCM}}\left(\prod_{(I,J)\in\mathcal{K}} \det M^{I,J}\right)\right),$$

The LCM is over disjoint multisubsets I and J of $[m]$ of size $\lfloor q/2 \rfloor$,

$$\text{and } M^{I,J} = \begin{pmatrix} \prod m_{i1} & \prod m_{i2} \\ \prod m_{j1} & \prod m_{j2} \end{pmatrix}.$$

Example: For $M_{4\times 2}$, we have $m = 4$.

Find all pairs (I, J) of disjoint $\lfloor q/2 \rfloor$-member multisubsets of $[4]$:

$$(\{1^a\}, \{2^b, 3^c, 4^d\}), \quad (\{2^b\}, \{1^a, 3^c, 4^d\}),$$
$$(\{3^c\}, \{1^a, 2^b, 4^d\}), \quad (\{4^d\}, \{1^a, 2^b, 3^c\}),$$
$$(\{1^a, 2^b\}, \{3^c, 4^d\}), \quad (\{1^a, 3^c\}, \{2^b, 4^d\}), \quad (\{1^a, 4^d\}, \{2^b, 3^c\}),$$

lcmd result

$$\text{lcmd}\left(M \otimes D(K_q)\right) = \text{lcm}\left((\text{lcmd}\, M)^{q-1}, \underset{\mathcal{K}}{\text{LCM}}\left(\prod_{(I,J)\in\mathcal{K}} \det M^{I,J}\right)\right),$$

The LCM is over disjoint multisubsets I and J of $[m]$ of size $\lfloor q/2 \rfloor$,

$$\text{and } M^{I,J} = \begin{pmatrix} \prod m_{i1} & \prod m_{i2} \\ \prod m_{j1} & \prod m_{j2} \end{pmatrix}.$$

Example: For $M_{4\times 2}$, $m = 4$. Consider $(I, J) = \left(\{1^a, 2^b\}, \{3^c, 4^d\}\right)$.

$$\text{Then, } M^{I,J} = \begin{pmatrix} m_{11}^a m_{21}^b & m_{12}^a m_{22}^b \\ m_{31}^c m_{41}^d & m_{32}^c m_{42}^d \end{pmatrix},$$

for all a, b, c, and d such that $a + b = c + d = \lfloor q/2 \rfloor$.

Bishop example

Example: When $P = ♗$ (bishop),

$$M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = M^T.$$

Applying the theorem,

- $(I, J) = \big(\{1^p\}, \{2^p\}\big)$.
- $\operatorname{lcmd}\big(M \otimes D(K_q)\big) = \operatorname{lcm}\Big(2^{q-1}, \operatorname*{LCM}_{p=2}^{\lfloor q/2 \rfloor} \big((-1)^p - 1^p\big)^{\lfloor q/2p \rfloor}\Big)$.
- The LCM term generates powers of 2 no larger than $2^{q/2}$.
- Hence, $\operatorname{lcmd}\big(M \otimes D(K_q)\big) = 2^{q-1}$.
- And our quasipolynomial period must divide 2^{q-1}

Back to that funny story. . .

Sketch of Kronecker theorem proof

$$\operatorname{lcmd}\left(M\otimes D(K_q)\right) = \operatorname{lcm}\left(\left(\operatorname{lcmd} M\right)^{q-1}, \underset{\mathcal{K}}{\operatorname{LCM}}\left(\prod_{(I,J)\in\mathcal{K}} \det M^{I,J}\right)\right),$$

Goal: Show that every $N_{l\times l}$ subdet. of $M \otimes D(K_q)$ divides RHS.

- Consider only N such that $\det(N) \neq 0$.
- N is a choice of l rows and l columns from $M \otimes D(K_q)$
- Same as a choice of l vertices and l edges from K_q, with up to m copies of each vertex and up to two copies of an edge.

$$M \otimes D(K_q) = \begin{pmatrix} m_{11}D(K_q) & m_{12}D(K_q) \\ m_{21}D(K_q) & m_{22}D(K_q) \\ \vdots & \vdots \\ m_{m1}D(K_q) & m_{m2}D(K_q) \end{pmatrix}$$

Sketch of Kronecker theorem proof

- When two rows correspond to the same vertex v, the rows contain the same entries except for different multipliers m_{ik}.

$$\begin{pmatrix} m_{11}D(K_q) & m_{12}D(K_q) \\ m_{21}D(K_q) & m_{22}D(K_q) \\ \vdots & \vdots \\ m_{m1}D(K_q) & m_{m2}D(K_q) \end{pmatrix} \quad \begin{pmatrix} \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ m_{i1} & -m_{i1} & 0 & \cdots & 0 & m_{i2} & 0 & \cdots & -m_{i2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ m_{j1} & -m_{j1} & 0 & \cdots & 0 & m_{j2} & 0 & \cdots & -m_{j2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \end{pmatrix}$$

- A vertex chosen three or more times would imply lin. dep.
- Simplify $\det N$ when a vertex is chosen twice. (This generates a factor of $\det M^{i,j}$).

$$\begin{pmatrix} \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ m_{i1} & -m_{i1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & m_{j2} & 0 & \cdots & -m_{j2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \end{pmatrix}$$

Sketch of Kronecker theorem proof

- Afterwards, every column has at most two entries.
- If a row (or column) has exactly one non-zero entry, expand.
- Then every row has exactly two non-zero entries as well.
- This matrix breaks down as a product of incidence matrices of weighted cycles, each of which basically contributes det $M^{I,J}$.

$$\begin{pmatrix} y_1 & 0 & 0 & 0 & 0 & -z_6 \\ -z_1 & y_2 & 0 & 0 & 0 & 0 \\ 0 & -z_2 & y_3 & 0 & 0 & 0 \\ 0 & 0 & -z_3 & y_4 & 0 & 0 \\ 0 & 0 & 0 & -z_4 & y_5 & 0 \\ 0 & 0 & 0 & 0 & -z_5 & y_6 \end{pmatrix}.$$

Not Queens

When $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$, again $\operatorname{lcmd}(M) = 2$.

Calculate $\det M^{I,J}$ for each pair (I, J):

- For example, when $I = \{3^c\}$ and $J = \{1^a, 2^b, 4^d\}$, and

$$M^{I,J} = \begin{pmatrix} 1^c & 1^c \\ 1^a 0^b 1^d & 0^a 1^b (-1)^d \end{pmatrix}.$$

  where $c = a + b + d = \lfloor q/2 \rfloor$.
- The only non-trivial case is when $a = b = 0$. Therefore $c = d = \lfloor q/2 \rfloor$ and $\det M^{I,J} = 0$ or -2.
- This implies that the LCM in the theorem divides 2^{q-1}.

We conclude that $\operatorname{lcmd}\left(M \otimes D(K_q)\right) = 2^{q-1}$.

Not Nightriders

Consider $M = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -2 \\ 2 & -1 \end{pmatrix}.$

- The submatrices

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

  have determinants -3, -4, and -5; hence $\text{lcmd}(M) = 60$.
- We have the same multisubsets of $[4]$ as before.
- $\det M^{I,J}$ is the same form in all cases: $\pm 2^u(2^{2\lfloor q/2 \rfloor - 2u} \pm 1)$, where u is a number between 0 and $\lfloor q/2 \rfloor$.

Not Nightriders

We conclude that

$$\operatorname{lcmd}\left(M \otimes D(K_q)\right) = \operatorname{lcm}\left(60^{q-1}, \operatorname*{LCM}_{\substack{1\le p\le q/2 \\ 0\le u\le p-1}} (2^{2p-2u} \pm 1)^{\lfloor q/2p \rfloor}\right).$$

When $q = 8$, $\operatorname{lcmd}\left(M \otimes D(K_8)\right) =$

$$\operatorname{lcm}\left(60^7, (4 \pm 1)^{\lfloor 8/2 \rfloor}, (16 \pm 1)^{\lfloor 8/4 \rfloor}, (64 \pm 1)^{\lfloor 8/6 \rfloor}, (256 \pm 1)^{\lfloor 8/8 \rfloor}\right)$$

$$= 60^7 \cdot 7 \cdot 13 \cdot 17^2 \cdot 257.$$

Not Nightriders

The first few values of q give the following numbers:

q	$\text{lcmd}\left(M \otimes D(K_q)\right)$	(factored)
2	60	60^1
3	3600	60^2
4	3672000	$60^3 \cdot 17$
5	220320000	$60^4 \cdot 17$
6	1202947200000	$60^5 \cdot 7 \cdot 13 \cdot 17$
7	72176832000000	$60^6 \cdot 7 \cdot 13 \cdot 17$
8	1892043474048000000	$60^7 \cdot 7 \cdot 13 \cdot 17^2 \cdot 257$
9	113522608442880000000	$60^8 \cdot 7 \cdot 13 \cdot 17^2 \cdot 257$
10	9522957531839431680000000	$60^9 \cdot 7 \cdot 11 \cdot 13 \cdot 17^2 \cdot 31 \cdot 41 \cdot 257$

View of our wandering from above

- Generalize n-Queens to q-Queens and beyond.
- Apply inside-out polytope theory to prove formula existence.
- Need to know the period; aim to find $\mathrm{lcmd}(A)$.
- On a rectangular board, $\mathrm{lcmd}(A) = \mathrm{lcmd}\left(M^T \otimes D(K_q)\right)$.
- Prove a theorem that applies to find $\mathrm{lcmd}\left(M \otimes D(K_q)\right)$.
- The theorem applies for $M_{2\times 2}$.

Open problems

- ► A better way to find the period? (LCMD is “bad”)
- ⊗ What goes wrong with more than two columns?
- ¿ Is a *formula* too much to ask?

$$\text{lcmd}\left(M \otimes D(K_q)\right) = \text{lcm}\left((\text{lcmd}\, M)^{q-1}, \underset{\mathcal{K}}{\text{LCM}} \left(\prod_{(I,J)\in\mathcal{K}} \det M^{I,J} \right) \right),$$

$p \cdot q$ When do two multivariate binomials have a common divisor?

$$(wx^2y - z^2u^2) \text{ and } (wy^3 - xz^2u)$$

Thank you

Slides available: `people.qc.cuny.edu/chanusa` > Talks

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  The incidence matrix of a complete graph.
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  Mathematical Analysis of a q-Queens Problem. In preparation.