

A q -Queens Problem


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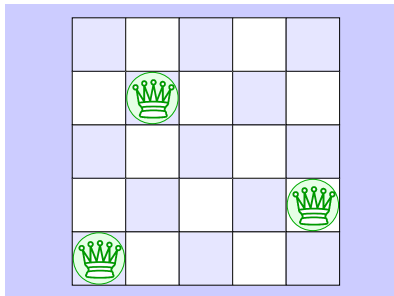
Joint work with

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When Queens Attack!

A **queen** is a chess piece that can move horizontally, vertically, and diagonally. 



- ▶ Two pieces are **attacking** when one piece can move to the other's square.
- ▶ A **configuration** is a placement of chess pieces on a chessboard.
- ▶ A configuration is **nonattacking** if no two pieces are attacking.

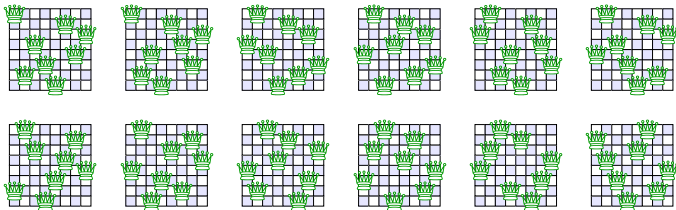
Question: How many nonattack'g queens MIGHT fit on a chessboard?

The 8-Queens Problem

Q: In how many ways

Q: Can you place 8 nonattacking queens on an 8×8 chessboard?

A: 92



The n -Queens Problem: Find a formula for the number of nonattacking configurations of n queens on an $n \times n$ chessboard.

n	1	2	3	4	5	6	7	8	9	10
#	1	0	0	2	10	4	40	92	352	724

From n -Queens to q -Queens

The n -Queens Problem:

nonatt. configs of n queens
on a $n \times n$ square board

A q -Queens Problem:

nonatt. configs of q pieces \mathbb{P}
on dilations of a polygonal board \mathcal{B}

- ▶ A number q .
of pieces in config.
- ▶ A piece \mathbb{P} .
A set of basic moves.
- ▶ A board \mathcal{B} .
A convex polygon
and its dilations.

A piece \mathbb{P} is defined by its moves $(c, d) \in \mathbf{M}$.

$$(x, y) \longrightarrow (x, y) + \alpha(c, d) \text{ for } \alpha \in \mathbb{Z}$$



Queen:

$$\mathbf{M} = \{(1, 0), (0, 1), (1, 1), (1, -1)\}$$



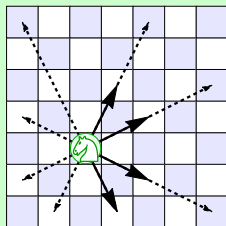
Bishop:

$$\mathbf{M} = \{(1, 1), (1, -1)\}$$



Nightrider:

$$\mathbf{M} = \{(1, 2), (1, -2), (2, 1), (2, -1)\}$$



From n -Queens to q -Queens

The n -Queens Problem:

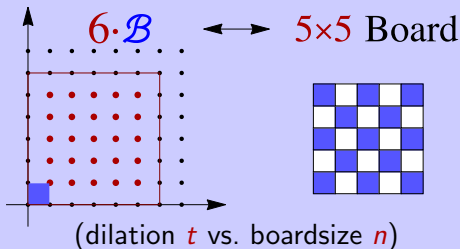
nonatt. configs of n queens
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A q -Queens Problem:

nonatt. configs of q pieces \mathbb{P}
on dilations of a polygonal board \mathcal{B}

- ▶ A number q .
of pieces in config.
- ▶ A piece \mathbb{P} .
A set of basic moves.
- ▶ A board \mathcal{B} .
A convex polygon and its dilations.

A **board** is the set of integral points on the *interior* of a dilation of a rational convex polygon $\mathcal{B} \subset \mathbb{R}^2$



A q -Queens Problem

Our Quest: Find a formula for the number of nonattacking configurations of q pieces \mathbb{P} inside dilations of \mathcal{B} .

Theorem: (CZ'05, CHZ'14)

Given q , \mathbb{P} , and \mathcal{B} , the number of nonattacking configurations of q pieces \mathbb{P} inside $t\mathcal{B}$ is a quasipolynomial function of t .

Definition: A **quasipolynomial** is a function $f(t)$ on $t \in \mathbb{Z}_+$ s.t. $f(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_0$, where each c_i is periodic in t .

Example. The number of ways to place two nightriders on an $n \times n$ chessboard is:

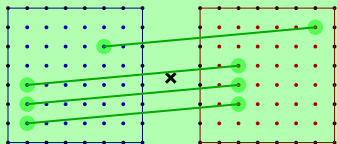
$$u_{\text{N}}(2; n) = \begin{cases} \frac{n^4}{2} - \frac{5n^3}{6} + \frac{3n^2}{2} - \frac{2n}{3} & \text{for even } n \\ \frac{n^4}{2} - \frac{5n^3}{6} + \frac{3n^2}{2} - \frac{7n}{6} & \text{for odd } n \end{cases}$$

Proof uses Inside-out polytopes

Two pieces \mathbb{P} in positions (x_i, y_i) and (x_j, y_j) inside $t\mathcal{B}$ are attacking if:

$$(x_i, y_i) - (x_j, y_j) = \alpha(c, d) \quad \overset{\text{move eqn.}}{\longleftrightarrow} \quad d(x_i - x_j) = c(y_i - y_j)$$

With **two** pieces, a **move equation** defines a **forbidden hyperplane** in $\mathcal{B}^2 \subset \mathbb{R}^4$.



Our quest becomes:
Count lattice points inside \mathcal{B}^q that avoid forbidden hyperplanes.

Inside-out polytope!
Apply theory of Beck and Zaslavsky.

- ▶ Answer is a quasipolynomial • degree $2q$ • $\text{vol}(\mathcal{B}^q) \rightsquigarrow$ initial term
- ▶ Inclusion-Exclusion for exact formula (later!)

Computing formulas experimentally

Restatement: The number of ways to place q \mathbb{P} -pieces inside a t dilation of \mathcal{B} is a quasipolynomial:

$$u_{\mathbb{P}}(q; t) = \left\{ \begin{array}{l} c_{2q,0} t^{2q} + \cdots + c_{1,0} t + c_{0,0} \quad t \equiv 0 \pmod{p} \\ c_{2q,1} t^{2q} + \cdots + c_{1,1} t + c_{0,1} \quad t \equiv 1 \pmod{p} \\ \vdots \\ c_{2q,p-1} t^{2q} + \cdots + c_{1,p-1} t + c_{0,p-1} \quad t \equiv p-1 \pmod{p} \end{array} \right\}$$

Consequence: If we know what the period is (or a bound), then **with enough data** we can solve for the $c_{i,j}$!

Gives a proof of correctness for $u_{\mathbb{P}}(q; t)$!

Enough data?

Let me introduce *Václav Kotěšovec*:

- ▶ Comprehensive Book
- ▶ Tables of Data
- ▶ Conjectured Formulas
- ▶ Essential check to our theory



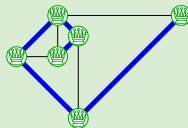
⚠ Collecting enough data is HARD for a large period. ⚠

Imp. Q. What is the period?

Thm. (qq.V) Bishops' period is 2.

Conj. (qq.IV, K.) Queens' period is $\text{lcm}(\{1, \dots, \text{fibonacci}_q\})$!?! 5:60

Discrete Fibonacci spiral!



Upper Bound (bad): LCM of denoms of facet/hyperpl. intersections.

Deriving formulas theoretically

Derive **exact formulas** for *leading coeffs* of quasipoly.:

Interior integer points **NOT** in the hyperplane arrangement is given by inclusion/exclusion on points **IN** the arrangement.

Calculate poset of
multiway intersections
of hyperplanes

Each corresponds to
placements of k attacking pieces

For each $\mathcal{U} \cap \mathcal{B}^q$, count
number of lattice points

We end up counting
number of ways k pieces attack

Apply **Inclusion/Exclusion** !

On a square board, $u_{\mathbb{P}}(q; n) = \frac{1}{q!} \sum_{\mathcal{U} \in \mathcal{L}(\mathcal{A}_{\mathbb{P}})} \mu(\hat{0}, \mathcal{U}) \alpha(\mathcal{U}; n) n^{2q-2k}$.

Examples of codimension 2 subspaces

**How might two attack equations interact?
And how do we count them?**

Four pieces

\mathbb{P}_1 attacks \mathbb{P}_2 on any slope.
 \mathbb{P}_3 attacks \mathbb{P}_4 on any slope.

[No interaction.]

(Count # ways two in a row)².

Three pieces

\mathbb{P}_1 attacks \mathbb{P}_2 on any slope.
 \mathbb{P}_2 attacks \mathbb{P}_3 on **another** slope.

[No restriction on \mathbb{P}_1 vs. \mathbb{P}_3 .]

Cases based on actual slopes.

Two pieces.

\mathbb{P}_1 attacks \mathbb{P}_2 on any slope.
 \mathbb{P}_1 attacks \mathbb{P}_2 on **another** slope.

[$\Rightarrow \mathbb{P}_1$ and \mathbb{P}_2 share a point.]

Count # of points on board.

Three pieces

\mathbb{P}_1 attacks \mathbb{P}_2 on any slope.
 \mathbb{P}_2 attacks \mathbb{P}_3 on **same** slope.

[$\Rightarrow \mathbb{P}_1$ and \mathbb{P}_3 also attack.]

Count # of ways three in a row.

✓ **Codim 3 for Partial Queens** $\mathbb{P} = \mathbb{Q}^{hk}$:

- explicit $u_{\mathbb{P}}(3; n)$
- leading 4 coeffs of $u_{\mathbb{P}}(q; n)$; period of 5–7.

A (not-very-useful) formula for n -Queens

Set $q = n$ to give the first closed-form formula for the n -Queens Problem:

Theorem

The number of ways to place n unlabelled copies of a rider piece \mathbb{P} on a square $n \times n$ board so that none attacks another is

$$\frac{1}{n!} \sum_{i=1}^{2n} n^{2n-i} \sum_{\kappa=2}^{2i} \binom{n}{\kappa} \sum_{\nu=\lceil \kappa/2 \rceil}^{\min(i, 2\kappa-2)} \sum_{[\mathcal{U}_{\kappa}^{\nu}]: \mathcal{U}_{\kappa}^{\nu} \in \mathcal{L}(\mathcal{A}_{\mathbb{P}}^{\infty})} \mu(\hat{0}, \mathcal{U}_{\kappa}^{\nu}) \frac{\bar{\gamma}_{i-\nu}(\mathcal{U}_{\kappa}^{\nu})}{|\text{Aut}(\mathcal{U}_{\kappa}^{\nu})|}.$$

This formula is very complicated but **it is** explicitly computable.

Brief Aside

I've never used so many variables!

- ▶ Blackboard letters: \mathbb{BNPQRZ}
- ▶ Bold letters: **abcdxyzILM β**
- ▶ Callig. letters: *ABCDEFGHIJKLMN~~OP~~QRSTUWXYZ*
- ▶ Greek letters: $\alpha\beta\gamma\delta\epsilon\zeta\theta\kappa\mu\nu\xi\pi\varphi\omega$ $\text{AB}\Delta\Gamma\text{H}\Lambda\P\Sigma\Psi$
- ▶ upper case: *ABCDEFGHIJKLMN~~OP~~QRSTU~~VW~~XYZ*
- ▶ lower case: *abcdefghijklmnopqrstuvwxy~~z~~*

(That's 99 variables!!! Plus the reuse of indices!)

What is next?

What Questions Are Interesting?




- ▶ **Fun test case** for Ehrhart Theory (lattice point) questions.
 - ▶ Period of quasipolynomial \neq LCM of denominators
- ▶ **Special pieces**
 - ▶ One-move riders show that period of quasip. depends on move
 - ▶ Other fairy pieces (see Kotěšovec's work)
- ▶ **Special boards**
 - ▶ Rook placement theory on other boards
 - ▶ Nice pieces on nice boards (Angles of 45, 90, 135 degrees)
- ▶ Determining all **subspaces** \mathcal{U} ; What is structure of posets?
- ▶ **Discrete Geometry**: Fibonacci spiral.

Thank you!

Slides available:

qc.edu/chanusa >
Research > Talks



-  Seth Chaiken, Christopher R. H. Hanusa, Thomas Zaslavsky.
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Electronic Journal of Combinatorics. **21**(3), #33, 28 pp. (2014)
-  Seth Chaiken, Christopher R. H. Hanusa, Thomas Zaslavsky.
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Journal of Algebraic Combinatorics. **41**(3). 619–642. (2015)
-  Seth Chaiken, Christopher R. H. Hanusa, Thomas Zaslavsky.
A q -Queens Problem. III. Partial queens. [arXiv:1402.4886](https://arxiv.org/abs/1402.4886)
A q -Queens Problem. V. The bishops' period. [arXiv:1405.3001](https://arxiv.org/abs/1405.3001)
A q -Queens Problem. IV. Bishops, queens, and n 'riders. [1609.00853](https://arxiv.org/abs/1609.00853)