EARTHQUAKES AND THURSTON'S BOUNDARY FOR THE TEICHMÜLLER SPACE OF THE UNIVERSAL HYPERBOLIC SOLENOID

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ABSTRACT. A measured laminations on the universal hyperbolic solenoid $\mathcal{S}$ is, by our definition, a leafwise measured lamination with appropriate continuity for the transverse variations. An earthquakes on the universal hyperbolic solenoid $\mathcal{S}$ is uniquely determined by a measured lamination on $\mathcal{S}$; it is a leafwise earthquake with the leafwise earthquake measure equal to the leafwise measured lamination. Leafwise earthquakes fit together to produce a new hyperbolic metric on $\mathcal{S}$ which is transversely continuous and we show that any two hyperbolic metrics on $\mathcal{S}$ are connected by an earthquake. We also establish the space of projective measured lamination $PML(\mathcal{S})$ as a natural Thurston-type boundary to the Teichmüller space $T(\mathcal{S})$ of the universal hyperbolic solenoid $\mathcal{S}$. The (baseleaf preserving) mapping class group $MCG_{BLP}(\mathcal{S})$ acts continuously on the closure $T(\mathcal{S})\cup PML(\mathcal{S})$ of $T(\mathcal{S})$. Moreover, the set of transversely locally constant measured laminations on $\mathcal{S}$ is dense in $ML(\mathcal{S})$.

1. INTRODUCTION

The universal hyperbolic solenoid $\mathcal{S}$ is the inverse limit of the system of all finite sheeted unbranched pointed covers of a compact surface of genus greater than 1. Sullivan [23] introduced the solenoid $\mathcal{S}$ as the "universal compact surface", i.e. the universal object in the category of finite unbranched covers of a compact surface. It turns out that $\mathcal{S}$ has a rich deformation theory, i.e. the Teichmüller space $T(\mathcal{S})$ is highly non-trivial. In fact, $T(\mathcal{S})$ is a first example of a Teichmüller space which is separable but not finite dimensional. (Recall that Teichmüller spaces of compact Riemann surfaces with possibly finitely many points removed are finite-dimensional complex manifolds, while Teichmüller spaces of geometrically infinite Riemann surfaces are non-separable infinite-dimensional complex Banach manifolds.)

Sullivan [23] started the study of the complex structure and the Teichmüller metric on the Teichmüller space $T(\mathcal{S})$ of the universal hyperbolic solenoid $\mathcal{S}$. The universal hyperbolic solenoid $\mathcal{S}$ has a transverse measure (unlike most of laminations) which is utilized in [22] to continue the investigation of the Teichmüller metric on $T(\mathcal{S})$. The results in [22] are used in [9] to show that generic points in $T(\mathcal{S})$ do not have Teichmüller-type Beltrami coefficient representatives which sharply contrasts the situation for finite surfaces (all points have Teichmüller type representatives) and for infinite surfaces (a dense, open subset of the Teichmüller space does). Thus, most points in $T(\mathcal{S})$ are not connected to the basepoint by a nice Teichmüller geodesic obtained by stretching horizontal and shrinking vertical

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foliation of a holomorphic quadratic differential on $\mathcal{S}$ unlike for Riemann surfaces (see [12] for the statement for infinite surfaces).

By results of Candel [6], a conformal structure on $\mathcal{S}$ contains a unique hyperbolic metric. This paper starts the investigation of the Teichmüller space $T(\mathcal{S})$ of the solenoid $\mathcal{S}$ using hyperbolic structures on $\mathcal{S}$. (An approach to studying the decorated Teichmüller space of the related punctured solenoid via hyperbolic structures is made in [17], [5]. This approach is specific to the punctured solenoid while the results in the paper hold for both the universal hyperbolic solenoid and the punctured solenoid.)

Our first result concerns the notion of an earthquake between two hyperbolic solenoids. An earthquake on a hyperbolic surface is a piecewise isometric bijective (not necessarily continuous) map from the hyperbolic surface to another hyperbolic surface. The support of an earthquake is a geodesic lamination (called the earthquake fault [24]) along which the quaking (i.e. the discontinuity) appears. The restriction to each stratum (i.e. a leaf of the support or a connected component of the complement of the support) of an earthquake is an isometry, and each stratum is moved to the left when viewed from any other stratum. Given an earthquake on a hyperbolic surface, there exists a unique transverse measure (called an earthquake measure) to its support [24] which determines the earthquake. (An earthquake measure is identified with a positive Radon measure, called a measured lamination, on the space of geodesics of a surface whose support is the support geodesic lamination and the measure of a bunch of geodesics in the support is given by the measure of an arc which intersects them and which does not intersect other geodesics in the support.) The main result concerning the earthquakes on hyperbolic surfaces is that any two hyperbolic metrics are connected by an earthquake [24].

We introduce a proper notion of an earthquake measure on the universal hyperbolic solenoid $\mathcal{S}$. An earthquake measure on $\mathcal{S}$ is an assignment of an earthquake measure on each leaf (isometric to a hyperbolic plane) of $\mathcal{S}$ such that the measures vary continuously for the transverse variations in an appropriate topology (see Definition 4.2 for more details). The support of leafwise earthquake measures are leafwise geodesic laminations which do not vary continuously for the transverse variations (see Example 4.3). However, the continuity of measures (in the appropriate Fréchet topology) guarantees that leafwise earthquakes on $\mathcal{S}$ piece together a new hyperbolic structure on $\mathcal{S}$ which is continuous for the transverse variations. Therefore, earthquake measures on $\mathcal{S}$ produce an earthquake of $\mathcal{S}$. We establish the transitivity statement for earthquakes on hyperbolic structures of the universal hyperbolic solenoid $\mathcal{S}$ analogous to the case of hyperbolic surfaces.

Theorem 5.1. *A measured lamination μ on a solenoid X with an arbitrary hyperbolic metric gives an earthquake map E_μ of X into another solenoid Y with a hyperbolic metric such that there exists a (differentiable) quasiconformal map $f : X \to Y$ whose extension to the boundary of leaves coincides with the extension of E_μ. Any two points in the Teichmüller space $T(\mathcal{S})$ of the universal hyperbolic solenoid $\mathcal{S}$ are connected by a unique earthquake along a measured lamination.*

To prove Theorem 5.1, we showed that if two quasisymmetric maps of the unit circle S^1 are close then their corresponding earthquake measures are close (in the

Fréchet topology). The opposite is false by an easy example. Therefore, the earthquake map from the space of bounded measured laminations in the unit disk onto the Teichmüller space is bijective but not a homeomorphism for the Fréchet topology on measured lamination. In the case of compact surfaces, the earthquake map is a homeomorphism (see [11]) when measured lamination are equipped with weak* topology (which is equivalent to the Fréchet topology for compact surfaces). We establish similar result for the solenoid $\mathcal{S}$.

Corollary 5.2. *The earthquake map which assigns to each bounded measured lamination on the universal hyperbolic solenoid $\mathcal{S}$ the corresponding marked hyperbolic solenoid is a homeomorphism between the space $ML(\mathcal{S})$ of bounded measured laminations and the Teichmüller space $T(\mathcal{S})$.*

Thurston [25],[10] introduced a natural boundary to the Teichmüller space of a compact surface by "adding at infinity" the space of projective measured laminations. The mapping class group acts continuously on the closure and there is a classification of its elements according to their action on the boundary. Bonahon [1] gave an alternative description of the Thurston's boundary to the Teichmüller space of a compact surface using the Liouville map which embeds the Teichmüller space into the space of measures on the space of geodesics of the surface. The boundary points at infinity are asymptotic rays to the image of the Teichmüller space. We used (see [21]) the idea of the Liouville embedding to give a Thurston-type boundary to the Teichmüller space of any (possibly geometrically infinite) Riemann surface. We extend this result to the Teichmüller space $T(\mathcal{S})$ of the universal hyperbolic solenoid $\mathcal{S}$.

Biswas, Nag and Mitra [4] introduced the direct limit of projective measured laminations on the compact surfaces as boundary at infinity of the direct limit of Teichmüller spaces of compact surfaces covering a fixed compact surface of genus at least two. Since $T(\mathcal{S})$ contains as a dense subset the above direct limit of Teichmüller spaces of compact surfaces, they remarked that the Thurston's boundary for $T(\mathcal{S})$ should be a completion of the union of the projective measured laminations on all compact surfaces. We give a proper analytical description of the Thurston's boundary answering their question about the completion. The main point is to properly define the continuity for the transverse variations of various spaces of measures and distributions on the space $\mathcal{G}(\mathcal{S})$ of geodesics on the universal hyperbolic solenoid $\mathcal{S}$. We establish this goal using the Fréchet topology on the ("enveloping") space of Hölder distributions $H(\mathcal{S})$ (see Section 6).

Theorem 6.2. *The Liouville map $\mathcal{L}_{\mathcal{S}} : T(\mathcal{S}) \to H(\mathcal{S})$ is a homeomorphisms onto its image. The set of asymptotic rays to $\mathcal{L}_{\mathcal{S}}(T(\mathcal{S}))$ is homeomorphic to the space of projective measured laminations on $\mathcal{S}$. The baseleaf preserving mapping class group $MCG_{BLP}(\mathcal{S})$ acts continuously on the closure $T(\mathcal{S}) \cup PML(\mathcal{S})$ of the Teichmüller space $T(\mathcal{S})$ of the universal hyperbolic solenoid $\mathcal{S}$.*

A lift of a measured lamination on a compact surface of genus at least two to the universal hyperbolic solenoid $\mathcal{S}$ is a measured lamination on $\mathcal{S}$. Such measured lamination is locally constant for the transverse variations and is called a transversely locally constant (TLC) measured lamination. We showed that an arbitrary measured lamination on $\mathcal{S}$ is the limit in the Fréchet topology of the TLC measured laminations.

Theorem 6.3. *The subset of all measured lamination on the universal hyperbolic solenoid $\mathcal{S}$ which are locally transversely constant is dense in the space of all measured laminations $ML(\mathcal{S})$ on $\mathcal{S}$ for the Fréchet topology.*

In Section 7, we introduce the space of compactly supported measured lamination $PML_0(\mathcal{S}_p)$ on the punctured solenoid $\mathcal{S}_p$ and extend the transitivity of earthquakes and Thurston's boundary for the punctured solenoid by replacing $ML(\mathcal{S})$ with $ML_0(\mathcal{S}_p)$.

2. Preliminaries

We recall several definitions: the universal hyperbolic solenoid, earthquakes of the unit disk, the Fréchet topology on the measures on the space of geodesics of the unit disk.

2.1. **The universal hyperbolic solenoid.**

Let (S, x) be a fixed compact surface of genus at least two with the basepoint $x \in S$. Consider all finite sheeted unbranched covers (S_i, x_i) by compact surfaces with basepoints such that the covering maps $\pi_i : S_i \to S$ satisfy $\pi_i(x_i) = x$. There is a natural partial ordering $\leq$ on the set of all such coverings. Namely, $(S_i, x_i) \leq (S_j, x_j)$ if there exists a finite sheeted unbranched covering map $\pi_{j,i} : S_j \to S_i$, $\pi_{j,i}(x_j) = x_i$, such that $\pi_i \circ \pi_{j,i} = \pi_j$. The set of all coverings is inverse directed, i.e. given two coverings of S there exists a third covering of S which is larger than the two (see [23], [16], [13], [22]). Sullivan [23] introduced the universal hyperbolic solenoid $\mathcal{S}$ as follows.

Definition 2.1. The *universal hyperbolic solenoid* $\mathcal{S}$ is the inverse limit (for the above partial ordering) of the directed system of all finite sheeted unbranched covers of a fixed compact surface of genus at least two.

The inverse limit $\mathcal{S}$ is independent of the base surface (i.e. two inverse limits with two base surfaces of genus at least two are homeomorphic). Thus it is called the universal hyperbolic solenoid $\mathcal{S}$.

We give an equivalent definition of the universal hyperbolic solenoid $\mathcal{S}$ [16]. Let G be a Fuchsian group uniformizing S, i.e. S is homeomorphic to $\mathbf{D}/G$, where $\mathbf{D}$ is the unit disk. Let G_n be the intersection of all subgroups of G with index at most n. Then G_n is of finite index in G. We define *profinite metric* on G (see [16]) by

$$dist(A, B) = e^{-n},$$

for $A, B \in G$, where $AB^{-1} \in G_n$ and $AB^{-1} \notin G_{n+1}$. The completion of G in the profinite metric *dist* is called the *profinite group completion* $\hat{G}$. The completion $\hat{G}$ is a compact, topological group homeomorphic to Cantor set. The group G is a dense subgroup of $\hat{G}$. We define an action of G on the product $\mathbf{D} \times \hat{G}$ by

$$A(z, t) := (A(z), tA^{-1}),$$

where $z \in \mathbf{D}$ and $t \in \hat{G}$, the action of A on $\mathbf{D}$ is just a Möbius map and A acts on $\hat{G}$ because G lies inside $\hat{G}$. The universal hyperbolic solenoid $\mathcal{S}$ is homeomorphic to the quotient $(\mathbf{D} \times \hat{G})/G$ (see [16]). From this description of $\mathcal{S}$, it is easy to see that $\mathcal{S}$ is a compact space which is locally homeomorphic to a 2-disk times Cantor set. Moreover, $\mathcal{S}$ fibers over $\mathbf{D}/G$ with fibers Cantor sets isomorphic to $\hat{G}$, and the restriction of the fiber map to each leaf is the universal covering of $S \equiv \mathbf{D}/G$. The same is true for any finite cover of S [15] by replacing G with the covering group.

Each $(\mathbf{D}\times\{t\})/G \subset \mathcal{S}$, for $t \in \hat{G}$, is a path component, called a *leaf* of $\mathcal{S}$. Each leaf is homeomorphic to the unit disk and it is dense in $\mathcal{S}$. Thus, each leaf is simply connected but the restriction of the topology on $\mathcal{S}$ to a leaf is not the standard topology on the unit disk. We define the universal cover of $\mathcal{S}$ by "straightening" the topology on leaves.

Definition 2.2. The *universal cover* for the universal hyperbolic solenoid $\mathcal{S}$ is given by

$$\pi : \mathbf{D} \times \hat{G} \to (\mathbf{D} \times \hat{G})/G,$$

where π is the quotient map for the action of G.

A *complex structure* on the universal hyperbolic solenoid $\mathcal{S}$ is a collection of local charts such that the transition maps are holomorphic in the disk direction and they vary continuously (for topology of uniform convergence) in the transverse (Cantor) direction of the local charts [23]. Complex structures on $\mathcal{S}$ are in one to one correspondence with conformal structures (which are continuous for the variations in the transverse direction) on $\mathcal{S}$ by the continuous dependence on the parameters of the solution of Beltrami equation. Any conformal structure on $\mathcal{S}$ contains a unique hyperbolic metric which is continuous for the variations in the transverse direction (see [6]). A hyperbolic metric (or a complex structure) on $\mathcal{S}$ which is transversely locally constant for some choice of charts on $\mathcal{S}$ is called a TLC hyperbolic metric (or a TLC complex structure) on $\mathcal{S}$ [23]. Any TLC hyperbolic metric (or complex structure) on $\mathcal{S}$ is obtained by lifting a hyperbolic metric (or a complex structure) from a finite cover of S to $\mathcal{S}$ [14]. Note that by identifying $\mathcal{S}$ with $(\mathbf{D} \times \hat{G})/G$ we fix a TLC complex structure on $\mathcal{S}$ coming from Riemann surface $\mathbf{D}/G$. A *(differentiable) quasiconformal map* $f : \mathcal{S} \to X$ from the fixed TLC complex solenoid $\mathcal{S}$ to an arbitrary complex solenoid X is a homeomorphism which is C^∞-differentiable in the disk direction in local charts and varies continuously in the transverse direction for C^∞-topology on C^∞-maps [23], [22]. We note that the use of differentiable quasiconformal maps as opposed to only quasiconformal maps is necessary in order for compositions of quasiconformal maps to be continuous in the transverse direction. We define the Teichmüller space $T(\mathcal{S})$ of the universal hyperbolic solenoid $\mathcal{S}$.

Definition 2.3. The *Teichmüller space* $T(\mathcal{S})$ of the universal hyperbolic solenoid $\mathcal{S}$ consists of all quasiconformal maps $f : \mathcal{S} \to X$ up to an equivalence. Two quasiconformal maps $f : \mathcal{S} \to X$ and $g : \mathcal{S} \to X_1$ are equivalent if there exists a conformal map $c : X \to X_1$ such that $g^{-1} \circ c \circ f : \mathcal{S} \to \mathcal{S}$ is homotopic to the identity. The equivalence class of the identity $id : \mathcal{S} \to \mathcal{S}$ is called the *basepoint* of $T(\mathcal{S})$.

2.2. **Earthquakes in the unit disk.** We define earthquakes of the unit disk $\mathbf{D}$ and recall their main properties. A *geodesic lamination* in the unit disk $\mathbf{D}$ is a closed subset of $\mathbf{D}$ which is foliated by geodesics for the hyperbolic metric on $\mathbf{D}$, or equivalently, it is a closed subset of the space $\mathcal{G}(\mathbf{D})$ of geodesics in $\mathbf{D}$ such that no two geodesics in the subset intersect in $\mathbf{D}$. Some familiar examples of geodesic laminations in $\mathbf{D}$ are: a set of finitely many non-intersecting geodesics in $\mathbf{D}$; a countable, discrete set of non-intersecting geodesics; a foliation of $\mathbf{D}$ by geodesics.

Definition 2.4. An *earthquake measure* on the unit disk is a positive Radon measure on the space of geodesic $\mathcal{G}(\mathbf{D})$ of the unit disk $\mathbf{D}$ whose support is a geodesic lamination.

Note that $\mathcal{G}(\mathbf{D})$ is homeomorphic to $(S^1 \times S^1 - diag)/\mathbb{Z}_2$ by mapping a geodesic in $\mathbf{D}$ to the unordered pair of its ideal endpoints in S^1. Thurston [24] introduced earthquakes as follows.

Definition 2.5. An *earthquake* $E : \mathbf{D} \to \mathbf{D}$ of the unit disk $\mathbf{D}$ is a bijective map which maps a fixed geodesic lamination λ (called the *support* of E) in $\mathbf{D}$ onto another geodesic lamination λ'. A geodesic from λ or a connected component of $\mathbf{D} - \lambda$ is called a *stratum* of E. The restriction of the earthquake E to each stratum is a hyperbolic isometry with the additional property that for any two strata A, B of E, the *comparison isometry*

$$E|_B \circ (E|_A)^{-1}$$

is a hyperbolic translation whose axis separates A from B, and which translates B to the left as seen from A.

Each earthquake $E : \mathbf{D} \to \mathbf{D}$ continuously extends to a homeomorphism of $\partial\mathbf{D} \equiv S^1$, which we denote by $E|_{S^1} : S^1 \to S^1$ [24]. An important theorem due to Thurston is that each orientation preserving homeomorphism of S^1 is obtained as the extension to S^1 of an earthquake [24]. Given an earthquake E of $\mathbf{D}$, there exists a unique corresponding earthquake measure μ on $\mathcal{G}(\mathbf{D})$ supported on λ determined by the following condition. Consider a subset of λ consisting of geodesics which intersect a closed arc I. Choose finitely many strata of E intersecting I. The μ measure of the subset is approximated by the sum of translation lengths between comparison isometries of adjacent strata (of the above chosen finitely many strata of E intersecting the arc I) when the distance between adjacent strata goes to zero [24]. An alternative description is to consider the measure μ to be a family of measures on arcs I in $\mathbf{D}$ which are invariant under homotopies of arcs preserving the leaves of λ. If an earthquake measure μ corresponds to an earthquake as above, we denote the corresponding earthquake by E_μ. Two homeomorphisms have the same corresponding earthquake measures if and only if they differ by a post-composition with a hyperbolic isometry of $\mathbf{D}$ [24].

We say that an earthquake measure μ is *bounded* if the *norm* $\|\mu\|$ satisfies

$$\|\mu\| := \sup_I \mu(I) < \infty,$$

where the supremum is over all geodesic arcs in $\mathbf{D}$ of length 1. If μ is a bounded earthquake measure, then there exists earthquake E_μ corresponding to μ. A homeomorphism $h : S^1 \to S^1$ is quasisymmetric if and only if the corresponding earthquake measure μ is bounded, where $E_\mu|_{S^1} = h$ [18], [19].

2.3. **The Fréchet topology.** We recall the definition of the Fréchet topology on the space of Hölder distributions $H(\mathbf{D})$ of the unit disk $\mathbf{D}$ from [20]. The space of bounded positive measures on $\mathcal{G}(\mathbf{D})$, and, in particular, the space of bounded earthquake measures on $\mathcal{G}(\mathbf{D})$ are subsets of $H(\mathbf{D})$.

The *Liouvile measure* L on $\mathcal{G}(\mathbf{D})$ is given by

$$L(K) := \iint_K \frac{d\alpha d\beta}{|e^{i\alpha} - e^{i\beta}|^2} \ ,$$

where $e^{i\alpha}, e^{i\beta} \in S^1$. A *box of geodesics* Q is the set of all geodesics in $(a,b)\times(c,d) \subset S^1\times S^1 - diag$, where $a,b,c,d \in S^1$ are different points given in the counter-clockwise order on S^1. Then

$$L(Q) = \log \Big|\frac{(a-c)(b-d)}{(a-d)(b-c)}\Big|,$$

and this formula can be used as an alternative definition of Liouville measure.

We fix $0 < \nu \leq 1$. The space of *ν-test functions $test(\nu)$* consists of all ν-Hölder continuous functions (φ, Q), $\varphi : \mathcal{G}(\mathbf{D}) \to \mathbb{R}$, whose support is in a box of geodesics Q with $L(Q) = \log 2$ such that $\|\varphi \circ \Theta_Q\|_\nu \leq 1$, where $\Theta_Q : (-1,-i)\times(1,i) \mapsto Q$ is a hyperbolic isometry and $\|\varphi\|_\nu := \max\{\sup_{\mathcal{G}(\mathbf{D})} |\varphi|, \sup_{(x,y)\neq(x_1,y_1)} \frac{|\varphi(x,y)-\varphi(x_1,y_1)|}{d((x,y),(x_1,y_1))^\nu}\}$, with d being the angle metric on $S^1 \times S^1$ (see [20],[21]).

The space of *Hölder distribution* $H(\mathbf{D})$ (see [21]) of the unit diks $\mathbf{D}$ consists of all linear functionals Ψ on the space of Hölder continuous functions $\varphi : \mathcal{G}(\mathbf{D}) \to \mathbb{R}$ with compact support such that

$$\|\Psi\|_\nu := \sup_{\varphi \in test(\nu)} |\Psi(\varphi)| < \infty$$

for all $0 < \nu \leq 1$. The Fréchet topology on $H(\mathbf{D})$ is defined using the family of ν-norms above. The topological vector space $H(\mathbf{D})$ is metrizable and one metric which gives the Fréchet topology is

$$dist(\Psi, \Psi_1) := \sum_{n=1}^{\infty} \frac{1}{n^2} \|\Psi - \Psi_1\|_{1/n}.$$

3. The convergence of measures in the unit disk

Denote by $\mathcal{G}_z$, for $z \in \mathbf{D}$, the set of geodesics in $\mathbf{D}$ which contain z. If $z, w \in \mathbf{D}$ then denote by $[z,w]$ the geodesic arc in $\mathbf{D}$ between z and w. If K is a subset of $\mathbf{D}$, denote by $\mathcal{G}_K$ the set of geodesics of $\mathbf{D}$ which intersect K.

We showed in [19] that a sequence of homeomorphisms of S^1 pointwise converges to a homeomorphism of S^1 if and only if the sequence of earthquake measures, corresponding to the sequence of homeomorphisms, converges to the earthquake measure of the limit. More precisely,

Proposition 3.1. [19] *Let μ, μ_i be uniformly bounded earthquake measures on $\mathbf{D}$, i.e. $\|\mu\|, \|\mu_i\| \leq M < \infty$. Then $\mu_i \to \mu$ in the weak* topology as $i \to \infty$ if and only if there exist normalizations of earthquake maps $E_{\mu_i}|_{S^1}, E_\mu|_{S^1}$ such that $E_{\mu_i}|_{S^1}(x) \to E_\mu|_{S^1}(x)$ for each $x \in S^1$, as $i \to \infty$. (Note that $E_{\mu_i}|_{S^1}, E_\mu|_{S^1}$ are well-defined up to the post-compositions by isometries and different normalizations correspond to different choices of isometries.)*

We consider a sequence of quasisymmetric maps converging to a quasisymmetric map in the quasisymmetric topology and show that the corresponding sequence of earthquake measures converges in the Fréchet topology.

Proposition 3.2. *Let $h_n = E_{\mu_n}|_{S^1}$ and $h = E_\mu|_{S^1}$ be quasisymmetric maps such that $h_n \to h$ as $n \to \infty$, in the quasisymmetric topology. Then $\|\mu_n - \mu\|_\nu \to 0$ as $n \to \infty$, for each $0 < \nu \leq 1$.*

Proof. Assume on the contrary that $\|\mu_n - \mu\|_\nu \geq m > 0$, for a fixed $0 < \nu \leq 1$. Thus, there exists $(\varphi_n, Q_n) \in test(\nu)$ such that $|\mu_n(\varphi_n) - \mu(\varphi)| \geq m > 0$, where $L(Q_n) = \log 2$. Without loss of generality, we assume that h_n, h fix $1, i, -1$. Let $Q_n' := h_n(Q_n)$ and $Q_n'' := h(Q_n)$. There exist unique hyperbolic isometries A_n, A_n', A_n'' of the unit disk $\mathbf{D}$ such that $A_n : Q_n \mapsto (1,i) \times (-1,-i)$, $A_n' : Q_n' \mapsto (1,i) \times (-1, q_n')$ and $A_n'' : Q_n'' \mapsto (1,i) \times (-1, q_n'')$, for unique $q_n', q_n'' \in S^1$.

Define $\bar{h}_n := A_n' \circ h_n \circ A_n^{-1}$ and $\bar{f}_n := A_n'' \circ h \circ A_n^{-1}$. Note that $\bar{h}_n : 1, i, -1, -i \mapsto 1, i, -1, q_n'$ and $\bar{f}_n : 1, i, -1, -i \mapsto 1, i, -1, q_n''$. The normalization of $\bar{h}_n$ and $\bar{f}_n$ implies that $\bar{h}_n \to \bar{h}$ and $\bar{f}_n \to \bar{f}$ pointwise on S^1, where $\bar{h}, \bar{f}$ are quasisymmetric maps as well. (This convergence is a consequence of pointwise convergence of a family of K-quasiconformal maps normalized to fix three points in $\hat{\mathbb{C}}$. Note that we can choose quasiconformal extensions of $\bar{h}_n, \bar{f}_n$ to have the same quasiconformal constant by using barycentric extension [7] in the interior and the exterior of the unit circle S^1.)

Consequently, $\bar{h}_n \circ \bar{f}_n^{-1} \to \bar{h} \circ \bar{f}^{-1}$ pointwise, as $n \to \infty$. Let Q be an arbitrary box with $L(Q) = \log 2$. Then $|L(A_n' \circ h_n \circ h^{-1} \circ (A_n'')^{-1}(Q)) - L(Q)| \to 0$ as $n \to \infty$ because $h_n \to h$ in the quasisymmetric topology and by the invariance of Liouville measure under hyperbolic isometries. Thus $\bar{h} \circ \bar{f}^{-1}$ preserves Liouville measure and fixes $1, i, -1$. Therefore $\bar{h} = \bar{f}$.

Let $\mu_n' := A_n^*(\mu_n)$ and let $\sigma_n := A_n^*(\mu)$. Then there exists a sequence $(\varphi_n', (1,i) \times (-1,-i)) \in test(\nu)$ such that

$$|\mu_n'(\varphi_n') - \sigma_n'(\varphi_n')| \geq m > 0. \tag{1}$$

Since μ_n' is the push forward by a hyperbolic isometry of μ_n then $\|\mu_n'\| = \|\mu_n\|$. Moreover, since μ_n are earthquake measures for h_n and h_n converges in the quasisymmetric topology, it follows that μ_n are uniformly bounded measures (and the same holds for μ_n'). The sequence σ_n is also uniformly bounded because it is the push forward of a single measure by hyperbolic isometries. Both sequences μ_n' and σ_n converge to bounded earthquake measures μ' and σ such that $\bar{h} = E_{\mu'}|_{S^1}$ and $\bar{f} = E_\sigma|_{S^1}$ by Proposition 3.1. By (1), we conclude that $\mu' \neq \sigma$. (To see that $\mu' \neq \sigma$, note that the weak* convergence on a fixed box $(1,i) \times (-1,-i)$ is equivalent to the uniform convergence with respect to all $(\varphi, (1,i) \times (-1,-i)) \in test(\nu)$. See [20] for details.) But this is a contradiction with $\bar{h} = \bar{f}$ by the uniqueness of earthquake measures [24]. □

We remark that the converse of Proposition 3.2 is not true. This is easily seen by an example. Take a fixed geodesic with a positive weight as one lamination. Take a convergent sequence in Fréchet topology to consists of geodesics sharing exactly one endpoint with the above geodesic and take the same positive weight. It is obvious that the extension of the earthquakes to S^1 corresponding to the sequence does not converge to the extension of the earthquake to S^1 corresponding to the limit in the quasisymmetric topology. Note that they do converge pointwise. This is in contrast with the statement in Proposition 3.1 which gives the equivalence. However, if we restrict ourselves to the earthquakes on compact surfaces then the equivalence holds. One of the main results in the next two sections is that the equivalence holds for the universal hyperbolic solenoid as well.

4. Measured laminations on the universal hyperbolic solenoid

Recall that a leaf of the universal hyperbolic solenoid $\mathcal{S}$ intersects any local chart countably many times. Each intersection is a 2-disk, which is called a *local leaf.* Given two local leaves of two global leaves, there exists an identification isometry of the global leaves given as follows. Since the hyperbolic metric in the local charts is continuous for the trivial (vertical) identification, we can choose two points sitting one above the other and two unit tangent vectors based at the points whose directions get vertically identified. (The two vectors are not necessarily vertically identified because the hyperbolic metrics are not constant in the transverse direction.) The isometric identification is uniquely determined by requiring to map one point onto the other other such that the unit vector is mapped onto the unit vector. The identification depends on the chart and the choice of two points while the choice of tangent vectors does not affect it.

Fix one local leaf l and consider a sequence of local leaves l_n approaching l. Suppose we choose two different isometric identifications $f_n : \tilde{l} \to \tilde{l}_n$ and $g_n : \tilde{l} \to \tilde{l}_n$ of the global leaves $\tilde{l}, \tilde{l}_n$ containing local leaves l, l_n (the identifications differ by the choice of points in the local leaves). Then $g_n^{-1} \circ f_n$ is an isometry of $\tilde{l}$ which converges to the identity as $n \to \infty$ because of the continuity in the transverse direction of the hyperbolic metrics. This implies that any two identifications of two global leaves differ by an isometry which is close to the identity when corresponding local leaves are close. Therefore, it makes sense to compare objects (preserved by isometries) on two nearby leaves as well as maps from leaves.

Definition 4.1. A (transversely continuous) *geodesic lamination* on the universal hyperbolic solenoid $\mathcal{S}$ is an assignment of a geodesic lamination to each leaf which is continuous (for Hausdorff distance between closed subset of $\mathcal{G}(\mathbf{D})$ defined using the angle metric d on $S^1 \times S^1$) with respect to the transverse variations given by each local chart as above.

Namely, for any local chart, we consider the isometric identifications as above. The geodesic laminations on global leaves can are mapped to the unit disk $\mathbf{D}$ by the identifications. Thus we obtain a map from the local transverse set (obtained by considering each local leaf in the chart as a point) to the space of geodesic laminations on the unit disk $\mathbf{D}$. We require that this map is continuous for the Hausdorff topology on the space of geodesic laminations.

This definition certainly seems in the spirit of transverse continuity of the hyperbolic metrics on $\mathcal{S}$. However, we introduce below measured laminations on $\mathcal{S}$ in terms of the continuity of measures. It turns out that the support of measured laminations on $\mathcal{S}$ are not geodesic laminations as above, even though the restriction to each leaf is a geodesic lamination.

Definition 4.2. A (transversely continuous) *measured lamination* μ on $\mathcal{S}$ is an assignment of a bounded measured lamination to each leaf of $\mathcal{S}$ such that it is continuous for the transverse variations with respect to Fréchet topology on the space of measured laminations on the unit disk.

Given a local chart $D \times T$, where D is a 2-disk and T a transverse Cantor set, the measured lamination μ on $\mathcal{S}$ gives a map $\mu : T \to ML_{bdd}(\mathbf{D})$ using the identifications of leaves induced by the local chart. In the above definition, we require that $\|\mu(t) - \mu(t')\|_\nu \to 0$ as $t' \to t$ for each $t \in T$ and for each $0 < \nu \leq 1$.

The definition of measured laminations on $\mathcal{S}$ does not specify the support. To give an example of a measured lamination on $\mathcal{S}$, fix a measured lamination σ on a compact surface S of genus at least two. Since each leaf of $\mathcal{S}$ is a universal cover of the compact surface S, we can lift σ to a measured lamination $\tilde{\sigma}$ on each leaf of $\mathcal{S}$. The lifts $\tilde{\sigma}$ are locally constant for the transverse variations in the local charts of a TLC hyperbolic metric coming from the hyperbolic metric on S. Thus $\tilde{\sigma}$ defines a geodesic lamination on $\mathcal{S}$.

As we mentioned above, there are measured laminations on $\mathcal{S}$ whose supports are not a geodesic laminations on $\mathcal{S}$. We give an example of a such measured lamination.

Example 4.3. We identify $\mathcal{S}$ with $(\mathbf{D}\times\hat{G})/G$, for a Fuchsian group G uniformizing a compact hyperbolic surface $S = \mathbf{D}/G$. We define a measured lamination $\tilde{\mu}$ on $\mathbf{D}\times\hat{G}$ which is invariant under G. Let G_i be a decreasing sequence of finite index normal subgroups of $G = G_1$ such that $\cap_{i=1}^{\infty} G_i = \{id\}$. We fix two simple closed curves γ_1, γ_2 in S which intersect in one point and we fix two lifts $\tilde{\gamma}_1, \tilde{\gamma}_2$ of γ_1, γ_2 in the universal cover $\mathbf{D}$ such that $|\tilde{\gamma}_1 \cap \tilde{\gamma}_2| = 1$. Denote by C_1, C_2 primitive hyperbolic translations in G whose axes are $\tilde{\gamma}_1, \tilde{\gamma}_2$ respectively. Let $C_1^{r_i}$ and $C_2^{t_i}$ be primitive elements in G_i. We further require that the group $\tilde{G}_{i+1}$ generated by $C_1^{r_i}, C_2^{t_i}$ and G_{i+1} is of index at least 3 in G_i.

Consider the cosets $a_0^i\tilde{G}_{i+1}, a_1^i\tilde{G}_{i+1}, \ldots, a_{k_i}^i\tilde{G}_{i+1}$, $a_0^i = id$, of $\tilde{G}_{i+1}$ in G. Since $[G_i : \tilde{G}_{i+1}] \geq 3$, there are at least two cosets different from $\tilde{G}_{i+1}$ which lie in G_i. Denote them by $a_1^i\tilde{G}_{i+1}, a_2^i\tilde{G}_{i+1} \subset G_i$. Then $(a_1^iG_{i+1}) \cdot (a_2^iG_{i+1})^{-1}$ are not of the form $C_j^kG_{i+1}$, for $j = 1$ or $j = 2$ and for some $k \in \mathbb{Z}$. To see this, first note that if $a_1^i(a_2^i)^{-1}$ is a power of C_1 or C_2 then it has to be a power of primitive elements $C_1^{r_i}$ or $C_2^{t_i}$ in G_i. (Otherwise $(a_1^iG_{i+1})\cdot(a_2^iG_{i+1})^{-1} \notin G_i/G_{i+1}$ which is a contradiction.) On the other hand, by our choice of $\tilde{G}_{i+1}$ and cosets $a_1^i\tilde{G}_{i+1}, a_2^i\tilde{G}_{i+1}$ we get that $a_1^i(a_2^i)^{-1}$ is not a power of primitive elements $C_1^{r_i}$ or $C_2^{t_i}$.

Let $\delta_{\tilde{\gamma}_i,t}$ denotes a unit mass measure on the space of geodesics of $\mathbf{D}\times\hat{G}$ supported on the geodesic $(\tilde{\gamma}_i, t) \subset \mathbf{D}\times\{t\}$, for $t \in \hat{G}$. We define

$$\tilde{\mu}' := \sum_i \Big(\sum_{t \in a_1^i\hat{G}_{i+1}} m_i\delta_{\tilde{\gamma}_1,t} + \sum_{t \in a_2^i\hat{G}_{i+1}} m_i\delta_{\tilde{\gamma}_2,t} \Big)$$

where $m_i > 0$, $m_i \to 0$ as $i \to \infty$ and $\hat{G}_{i+1} < \hat{G}$ is the profinite completion of G_{i+1}. The measured lamination $\tilde{\mu}'$ is varying continuously in the transverse direction for the Fréchet topology on measured laminations of the unit disk. The continuity is immediate at any $t \in \hat{G}$ because $\tilde{\mu}'$ is locally constant.

We define

$$\tilde{\mu} := \sum_{A \in G} A^*(\tilde{\mu}').$$

Then $\tilde{\mu}$ is invariant under the action of G. We first show that the support of $\tilde{\mu}$ on each leaf $\mathbf{D}\times\{t\}$, $t\in\hat{G}$, is a geodesic lamination. Let ω be a fundamental polygon for the action of G in $\mathbf{D}$. Then $\omega\times\hat{G}$ is a fundamental set for the action of G on $\mathbf{D}\times\hat{G}$. It is enough to show that the support of $\tilde{\mu}$ has no self-intersections on $\omega\times\hat{G}$. Note that the support of $\tilde{\mu}'$ has no self-intersections because the cosets of $\hat{G}$ which contain copies of $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ in the support are chosen to be disjoint. The only possibility for the support of $\tilde{\mu}$ to have a self-intersection is if a coset $a_1^i\hat{G}_{i+1}$ is mapped onto the coset $a_2^i\hat{G}_{i+1}$ by C_1^k for some $k\in\mathbb{Z}$, and similar for C_2. This is impossible by our choice of cosets.

We claim that $\tilde{\mu}$ is continuous for the transverse variations in the Fréchet topology on the space of measured laminations of the unit disk $\mathbf{D}$. We first can assume that different lifts of γ_j, $j=1,2$, in $\mathbf{D}$ do not belong to a single box of geodesics Q with $L(Q)=\log 2$ by appropriately choosing the group G. If we show continuity in this case, the result follows because the convergence of measured laminations in the Fréchet topology is independent of the hyperbolic metric. We already concluded that $\tilde{\mu}'$ is continuous for the transverse variations. By taking the push-forward of $\tilde{\mu}'$ by G, we add some extra support of $\tilde{\mu}$ intersecting fundamental set $\omega\times\hat{G}$. The extra support is obtained by adding $(\tilde{\gamma}_1,t)$ for $t\in C_1^ka_1^i\hat{G}_{i+1}$, $k\in\mathbb{Z}$ and $i=1,2,\ldots$, and by adding $(\tilde{\gamma}_2,t)$ for $t\in C_2^ka_2^i\hat{G}_{i+1}$, $k\in\mathbb{Z}$ and $i=1,2,\ldots$. It is obvious that the restriction of $\tilde{\mu}$ to the part which intersects $\omega\times\hat{G}$ is continuous for the transverse variations at any $t\in\hat{G}-\{id\}$, similar to $\tilde{\mu}'$. The continuity at $t=id$ follows because $m_i\to 0$. Our assumption that the orbit of $\tilde{\gamma}_i$ does not contain two geodesic which lie in a box Q with $L(Q)=\log 2$ implies the continuity of $\tilde{\mu}$.

The measured lamination $\tilde{\mu}$ on $\mathbf{D}\times\hat{G}$ descends to a measured lamination μ on the universal hyperbolic solenoid $\mathcal{S}$. The continuity of μ for the transverse variations follows by the continuity of $\tilde{\mu}$. It is clear that the support of μ is not a geodesic lamination on $\mathcal{S}$ as in Definition 4.1 because it is not a closed set. Moreover, the closure in $\mathcal{S}$ of the support of μ is not a geodesic lamination because on the baseleaf it consists of the full preimage of the two intersecting geodesics γ_1 and γ_2 on the closed surface S. □

5. Earthquake theorem

We show that any two points in the Teichmüller space $T(\mathcal{S})$ of the universal hyperbolic solenoid $\mathcal{S}$ are connected by an earthquake. We first need to recall certain facts from [21] about arbitrary points in $T(\mathcal{S})$.

A TLC solenoid $\mathcal{S}$ is homeomorphic to $(\mathbf{D}\times\hat{G})/G$, where $\mathbf{D}$ is the unit disk. The space $\mathbf{D}\times\hat{G}$ is considered as a universal cover of $\mathcal{S}$. Denote by $\pi:\mathbf{D}\times\hat{G}\to\mathcal{S}$ the covering map. The action of G is given by

$$A(z,t):=(A(z),tA^{-1}),$$

where A can be considered as an element of $\hat{G}$.

A point in $T(\mathcal{S})$ is given by a (differentiable) quasiconformal map $f:\mathcal{S}\to X$, where X is the universal hyperbolic solenoid with an arbitrary hyperbolic metric (not necessarily TLC). We introduced (see [21]) the universal (hyperbolic) cover to X and the covering group as follows. The action by G does not introduce identifications to the set $\{0\}\times\hat{G}$, $0\in\mathbf{D}$. Consider a local chart $D\times T$ for X

which contains $f(\pi(\{0\} \times \hat{G}))$ as a vertical set, where $T \equiv \hat{G}$ and D is a 2-disk with center at 0. To fix the notation, we assume that $f(\pi(\{0\} \times \hat{G}))$ corresponds to $\{0\} \times T \subset D \times T$ in the local chart. We fix unit tangent vectors at the points $f(\pi(\{0\} \times \hat{G}))$ corresponding under the chart map to the unit tangent vectors at the points $\{0\} \times T$ along the positive axis in the chart $D \times T$.

The universal cover for X is, by the definition, $\mathbf{D} \times T$ and the covering map $\pi_X : \mathbf{D} \times T \to X$ is given by isometrically mapping each $\mathbf{D} \times \{t\}$ onto the leaf containing $f(\pi(0,t))$ such that the origins are mapped onto the origins and the unit tangent vectors along positive axes are mapped onto the unit tangent vectors along the positive axes when considered in the chart $D \times T$. The map $f : \mathcal{S} \to X$ lifts to a quasiconformal map

$$\tilde{f} : \mathbf{D} \times \hat{G} \to \mathbf{D} \times T$$

of the universal covers.

The action of G is conjugated by $\tilde{f}$ to an action of a group G_X on $\mathbf{D} \times T$. An element A of G acts on $\mathbf{D} \times \hat{G}$ by $A(z,t) = (A(z), tA^{-1})$, namely it acts on the unit disk component by hyperbolic isometry and it shifts the leaf $\mathbf{D} \times \{t\}$ onto the leaf $\mathbf{D} \times \{tA^{-1}\}$. In particular, the action on the disk coordinate is independent of the leaf (the second coordinate). We define a covering transformation A_f for X by the formula

$$(A_f \circ \tilde{f})(z,t) = (\tilde{f} \circ A)(z,t).$$

An equivalent definition for A_f is

$$A_f(z,t) := (\pi_X)^{-1}(\pi_X(z,t), tA^{-1}),$$

where $\pi_X(z,t) \in \mathcal{S}$ and $\pi_X(\cdot, tA^{-1})$ stands for the inverse of the covering map restricted to $\mathbf{D} \times \{tA^{-1}\}$. (Recall that the covering map when restricted to each leaf is an isometry for the hyperbolic metric on leaves.) Thus A_f is an isometry on each leaf of $\mathbf{D} \times T$, but it varies with leaves. The group of covering maps for the hyperbolic solenoid X is denoted by $G_X := \tilde{f} G \tilde{f}^{-1}$. (If all A_f in a finite index subgroup of G_X are constant in the transverse direction then X has a TLC hyperbolic metric.)

We show that earthquakes are transitive in $T(\mathcal{S})$.

Theorem 5.1. *A measured lamination μ on a solenoid X with an arbitrary hyperbolic metric gives an (leafwise) earthquake map E_μ of X into another solenoid Y with hyperbolic metric such that there exists a (differentiable) quasiconformal map $f : X \to Y$ whose extension to the boundary of leaves coincides with the extension of E_μ. Any two points in the Teichmüller space $T(\mathcal{S})$ of the universal hyperbolic solenoid $\mathcal{S}$ are connected by a unique earthquake along a measured lamination on $\mathcal{S}$.*

Proof. We first show that an earthquake map along a measured lamination μ on the hyperbolic solenoid X gives another hyperbolic solenoid Y. We recall that μ is an assignment of bounded measured laminations to the leaves of X such that it varies continuously for the transverse variations.

Let $f : \mathcal{S} \to X$ be a differentiable quasiconformal map, where $\mathcal{S} = (\mathbf{D} \times \hat{G})/G$. Recall the universal cover $\pi_X : \mathbf{D} \times T \to X$ and lift μ to a measured lamination

$\tilde{\mu}$ which varies continuously for the transverse variations. In other words, $\tilde{\mu}: T \to ML_{bdd}(\mathbf{D})$ is continuous for the Fréchet topology on the space $ML_{bdd}(\mathbf{D})$ of bounded measured laminations of $\mathbf{D}$ and it satisfies the invariance under the action of G_f, i.e.

$$(A_f(t))^*(\tilde{\mu}(t)) = \tilde{\mu}(tA^{-1})$$

for all $A_f \in G_f$, where $A_f(t)$ is the hyperbolic isometry of $\mathbf{D}$ obtained by restricting A_f to a map from $\mathbf{D} \times \{t\}$ onto $\mathbf{D} \times \{tA^{-1}\}$.

We consider a family of earthquakes $E_{\tilde{\mu}(t)}: \mathbf{D} \to \mathbf{D}$, for $t \in T$, and we normalize them to fix 1, i and -1 on the unit circle $S^1 = \partial\mathbf{D}$. We first show that they induce a family of quasisymmetric maps of $S^1 \times T \equiv \partial(\mathbf{D}) \times T$ onto itself which conjugate G_f onto another group of leafwise isometries. Let $h_t := E_{\tilde{\mu}(t)}|_{S^1}$, where $h_t : 1, i, -1 \mapsto 1, i, -1$. By the above invariance of $\tilde{\mu}$ under the action of A_f and by the fact that an earthquake is determined, up to post-composition with a hyperbolic isometry, by its measure [24], we get

$$h_{tA^{-1}} \circ A_f(t) = A_*(t) \circ h_t,$$

where $A_*(t)$ is a hyperbolic isometry between $\mathbf{D} \times \{t\}$ and $\mathbf{D} \times \{tA^{-1}\}$ defined by the equation. To each $A_f \in G_f$ we assign such A_* which is a hyperbolic isometry between leaves. The maps A_*, for all $A_f \in G_f$, form a group G_* isomorphic to G_f under the conjugation by h.

We claim that $A_*(t)$ is continuous in t for the standard topology on the space of hyperbolic isometries of $\mathbf{D}$. To see this, it is enough to show that the images of three fixed point on S^1 vary continuously in t. By the above equation, we get

$$A_*(t) = h_{tA^{-1}} \circ A_f(t) \circ h_t^{-1}.$$

Since $\tilde{\mu}(t)$ varies continuously in t and since $h_t, h_{tA^{-1}}$ are properly normalized earthquakes, we conclude that $h_t, h_{tA^{-1}}$ are continuous in t for the topology of pointwise convergence by [19, Proposition 3.3]. By our assumption, $A_f(t)$ is continuous in t. Thus $A_*(t)$ is continuous in t.

We need to show that the quotient $(\mathbf{D}\times T)/G_*$ is quasiconformally equivalent to X, i.e. there exists a homeomorphism $g : X \to (\mathbf{D}\times T)/G_*$ which is a differentiable quasiconformal map on each leaf and which varies continuously in the transverse direction (in the C^∞-topology). Let $g_t = ex(h_t : \mathbf{D} \to \mathbf{D})$ be barycentric extension of $h_t : S^1 \to S^1$ (see [7] for the definition and properties of barycentric extension). Recall that the family h_t, $t \in T$, is continuous in the pointwise convergence topology. Then the family g_t of barycentric extensions is continuous for the C^∞-topology of C^∞-maps (over compact subsets of $\mathbf{D}$) by [7].

We claim that f_t is also continuous in the parameter t for the quasiconformal topology. (Note that the fact that the earthquake measures converge in the Fréchet topology does not imply that the extension of earthquakes to S^1 converge in the quasisymmetric topology by the example in Section 3. At this point we strongly use compactness of the solenoid X.) Recall that G_X has a compact fundamental set for the action on $\mathbf{D}\times T$ [22] (given by the image under $\tilde{f}$ of the fundamental set $\omega \times \hat{G}$ for the action of G on $\mathbf{D} \times \hat{G}$). Thus the Beltrami coefficients of the family g_t are continuous in t for the supremum norm over the fundamental set of X. By the invariance of the quasisymmetric family h_t and by the conformal naturallity of barycentric extension [7], we obtain

$$g_{tA^{-1}} \circ A_f(t) = A_*(t) \circ g_t,$$

for $t \in T$. This invariance under G_f and the continuity of Beltrami coefficients of g_t on the fundamental domain of G_f implies that the Beltrami coefficients of g_t are continuous in t for the essential supremum norm on the unit disk. Thus we obtained a differentiable quasiconformal homeomorphism $\tilde{g} : \mathbf{D} \times T \to \mathbf{D} \times T$, $\tilde{g}(\cdot, t) := g_t(\cdot)$, which conjugates G_f onto G_*. Consequently it projects onto a quasiconformal homeomorphism $g : X \to (\mathbf{D} \times T)/G_*$. Thus the earthquake E_μ defines a new hyperbolic solenoid $Y := (\mathbf{D} \times T)/G_*$ which is the image of X. By its definition, the boundary values of each g_t agree with $E_{\tilde{\mu}(t)}|_{S^1}$. This finishes the proof of the first part of the Theorem.

It remains to show that any two points $[f : \mathcal{S} \to X], [g : \mathcal{S} \to Y]$ are connected by an earthquake along a measured lamination on X. In other words, we need to find a measured lamination μ on X such that E_μ maps X onto Y and that the extensions of E_μ to the boundaries of leaves are equal to the extensions of $g \circ f^{-1}$. We lift the maps f and g to the maps $\tilde{f} : \mathbf{D} \times T \to \mathbf{D} \times T$ and $\tilde{g} : \mathbf{D} \times T \to \mathbf{D} \times T$ of the universal covers of X and Y. Let $h_t := \tilde{g} \circ \tilde{f}^{-1}|_{\mathbf{D} \times \{t\}}$. Note that the family of quasisymmetric maps h_t, $t \in T$, is continuous in t for the quasisymmetric topology. By Thurston's earthquake theorem for the unit disk [24], there exists a measured lamination $\tilde{\mu}(t)$ such that $E_{\tilde{\mu}(t)}|_{S^1} = h_t$. Moreover, $\tilde{\mu}(t)$ is a bounded earthquake measure on $\mathbf{D}$. Since h_t vary continuously, we get that $\tilde{\mu}(t)$ vary continuously in t for the Fréchet topology by Proposition 3.2. The family h_t satisfies invariance properties with respect to G_X. Therefore, by the uniqueness of earthquake measures [24], the family of corresponding earthquake measures $\tilde{\mu}(t)$ also satisfies invariance properties. Thus it descend to the desired earthquake measure μ on X. □

We recall that Proposition 3.2 states that if quasisymmetric maps are close (in the quasisymmetric topology) then corresponding earthquake measures are close (in the Fréchet topology). The converse is false in general. However, we showed above that the compactness of the universal hyperbolic solenoid $\mathcal{S}$ forces the continuity of quasisymmetric maps on nearby leaves obtained by earthquaking along transversely continuous measured laminations. The proof extends along the same lines to show that if two measured laminations on the solenoid $\mathcal{S}$ are close in the Fréchet topology then the extension of earthquake maps to the boundary leaves are close in the quasisymmetric topology. We obtained

Corollary 5.2. *The earthquake map which assigns to each bounded measured lamination on the universal hyperbolic solenoid $\mathcal{S}$ the corresponding marked hyperbolic solenoid is a homeomorphism between the space $ML(\mathcal{S})$ of bounded measured laminations and the Teichmüller space $T(\mathcal{S})$.*

6. Thurston's Boundary for $T(\mathcal{S})$

We recall the definition of the Liouville map $Ł : T(\mathbf{D}) \to H(\mathbf{D})$ from the universal Teichmüller space $T(\mathbf{D})$ to the space of Hölder distributions $H(\mathbf{D})$ of the unit disk $\mathbf{D}$. (Liouville map first appears in [1] in the case of the Teichmüller space of a compact Riemann surface and it is used in [21] to introduce Thurston-type boundary to the universal Teichmüller space $T(\mathbf{D})$.)

The *universal Teichmüller space* $T(\mathbf{D})$ is the set of all quasisymmetric maps $h : S^1 \to S^1$ which fix $1, i, -1$. The topology on $T(\mathbf{D})$ is defined by requiring that two quasisymmetric maps are close if there exist their quasiconformal extensions to $\mathbf{D}$ whose Beltrami coefficients are close in the essential supremum norm on $\mathbf{D}$.

The *Liouville map* $\mathcal{L} : T(\mathbf{D}) \to H(\mathbf{D})$ is defined by taking the pull-back

$$\mathcal{L}(h) := h_*(L)$$

of the Liouville measure L by the quasisymmetric maps $h \in T(\mathbf{D})$. The Liouville map $\mathcal{L}$ is a homemorphism of $T(\mathbf{D})$ onto its image; the image $\mathcal{L}(T(\mathbf{D})) \subset H(\mathbf{D})$ is closed and unbounded (see [21]). An *asymptotic ray* to $L(T(\mathbf{D}))$ is a path $t\Psi$, $t > 0$ and $\Psi \in H(\mathbf{D})$, such that there exists a path α_t, $t > 0$, in $T(\mathbf{D})$ with

$$\frac{1}{t}\alpha_t \to \Psi,$$

as $t \to \infty$. Each positive ray through the origin intersects the image $L(T(\mathcal{G}))$ in at most one point. Therefore, under the projection of the vector space $H(\mathbf{D})$ to the unit sphere (in $H(\mathbf{D})$, for a fixed ν-norm), the set Ł$(T(\mathbf{D}))$ is mapped homeomorphically and its boundary corresponds to the asymptotic rays. Thus, we consider asymptotic rays to $\mathcal{L}(T(\mathbf{D}))$ as a natural boundary to $T(\mathbf{D})$. In [21], we characterized the boundary points of the universal Teichmüller space $T(\mathbf{D})$ as all asymptotic rays along bounded measured laminations. Namely,

Theorem 6.1. [21] *The Liouville map $\mathcal{L} : T(\mathbf{D}) \to H(\mathbf{D})$ is a homeomorphism onto its image and $\mathcal{L}(T(\mathbf{D}))$ projects homeomorphically to the unit sphere. The boundary of $T(\mathbf{D})$ is identified by the above embedding with the space of bounded projective measured laminations $PML_{bdd}(\mathbf{D})$. The (quasiconformal) mapping class group $QMCG(\mathbf{D})$ acts continuously on the closure $T(\mathbf{D}) \cup PML_{bdd}(\mathbf{D})$ of the universal Teichmüller space $T(\mathbf{D})$.*

We introduce a Thurston-type boundary to the Teichmüller space $T(\mathcal{S})$ of the universal hyperbolic solenoid $\mathcal{S}$. The space of geodesics on $\mathcal{S} = (\mathbf{D} \times \hat{G})/G$ is naturally identified with the G-orbits of points in $(S^1 \times S^1 - diag) \times \hat{G}$ given by lifting a single geodesic on a leaf of $\mathcal{S}$ to the universal cover $\mathbf{D} \times \hat{G}$. Since each leaf of $\mathcal{S}$ is isometric to the hyperbolic plane, it supports the Liouville measure on the space of its geodesics. Thus $\mathcal{S}$ has a leafwise Liouville measure which lifts to a leafwise measure, called the *leafwise Liouville measure* L_{leaf}, on the space of geodesics of the universal cover $\mathbf{D} \times \hat{G}$.

A *(leafwise) Hölder distribution* Ψ for the universal hyperbolic solenoid $\mathcal{S}$ is a family of Hölder distributions $\Psi_t \in H(\mathbf{D})$, for $t \in \hat{G}$, which are invariant under the action of G, i.e.

$$\Psi_{tA^{-1}}(\varphi \circ A^{-1}) = \Psi_t(\varphi),$$

where φ is a Hölder continuous function with compact support on the space of geodesics $\mathcal{G}(\mathbf{D})$, and which vary continuously in t for the Fréchet topology, i.e.

$$\|\Psi_t - \Psi_{t_1}\|_\nu \to 0,$$

as $t_1 \to t$ for each $t \in \hat{G}$ and for each $0 < \nu \leq 1$.

The ν-*norm* of a leafwise Hölder distribution Ψ is given by

$$\|\Psi\|_\nu := \sup_{t\in\hat{G},\varphi\in test(\nu)} |\Psi_t(\varphi)|,$$

for $0 < \nu \leq 1$, where $test(\nu)$ is the set of ν-test functions on $\mathcal{G}(\mathbf{D})$. If $\|\Psi\|_\nu < \infty$ for all $0 < \nu \leq 1$ then Ψ is called *bounded leafwise Hölder distribution*. The space of all (bounded) leafwise Hölder distributions for the universal hyperbolic solenoid $\mathcal{S}$ is denoted by $H(\mathcal{S})$.

Let $[f : \mathcal{S} \to X] \in T(\mathcal{S})$ be an arbitrary point and denote by $\tilde{f} : \mathbf{D}\times\hat{G} \to \mathbf{D}\times T$ the lift of f to the universal cover. Let $h : S^1 \times \hat{G} \to S^1 \times T$ be the leafwise quasisymmetric extension of $\tilde{f}$ to the boundary of leaves. We define the *Liouville map* $\mathcal{L}_{\mathcal{S}} : T(\mathcal{S}) \to H(\mathcal{S})$ for the universal hyperbolic solenoid $\mathcal{S}$ by

$$\mathcal{L}_{\mathcal{S}}([f]) = h_*(L_{leaf}).$$

(Note that bounded measures on $\mathcal{G}(\mathbf{D})$ are in $H(\mathbf{D})$ and that a pull-back by a quasisymmetric of the Liouville measure is bounded [21]. Thus the image of the Liouville map is in $H(\mathbf{D})$ and the leafwise statement for the universal hyperbolic solenoid immediately follows.)

We show that the Liouville map is an embedding and that the natural boundary (i.e. the set of asymptotic rays) is homeomorphic to the space of projective measured laminations on $\mathcal{S}$.

Theorem 6.2. *The Liouville map $\mathcal{L}_{\mathcal{S}} : T(\mathcal{S}) \to H(\mathcal{S})$ is a homeomorphisms onto its image. The set of asymptotic rays to $\mathcal{L}_{\mathcal{S}}(T(\mathcal{S}))$ is homeomorphic to the space of projective measured laminations on $\mathcal{S}$. The baseleaf preserving mapping class group $MCG_{BLP}(\mathcal{S})$ acts continuously on the closure $T(\mathcal{S}) \cup PML(\mathcal{S})$ of the Teichmüller space $T(\mathcal{S})$ of the universal hyperbolic solenoid $\mathcal{S}$.*

Proof. The Liouville map $Ł_{\mathcal{S}}$ is assigning to any $[f] \in T(\mathcal{S})$ the pull-backs of the leafwise Liouville measures on $\mathcal{G}(\mathbf{D}\times\hat{G})$ by the extensions h to $S^1\times\hat{G}$ of the lift $\tilde{f} : \mathbf{D}\times\hat{G} \to \mathbf{D}\times\hat{G}$. The continuity of $Ł : T(\mathbf{D}) \to H(\mathbf{D})$ implies that $(h_t)_*(L)$ is continuous in t for the Fréchet topology. Thus Ł maps $T(\mathcal{S})$ into $H(\mathcal{S})$. Recall that $T(\mathcal{S})$ embeds in the universal Teichmüller space $T(\mathbf{D})$ by restricting the map $f : \mathcal{S} \to X$ to the baseleaf of $\mathcal{S}$ [14]. Denote by $T_{restr.}(\mathbf{D})$ the image of the embedding. Also, since the baseleaf is dense in $\mathcal{S}$, the restriction to the beaseleaf of the pull-back of the leafwise Liouville measure completely determines the measure. Therefore, the restriction of the Liouville map $\mathcal{L}_{\mathcal{S}}$ to the baseleaf($\equiv \mathbf{D}$) $\mathcal{L} : T_{restr.}(\mathbf{D}) \to H(\mathbf{D})$ completely determines the map. Since $Ł : T(\mathbf{D}) \to H(\mathbf{D})$ is a homeomorphism onto its image, it follows that $\mathcal{L} : T_{restr.}(\mathbf{D}) \to H(\mathbf{D})$ is also a homeomorphism onto its image. Therefore, $\mathcal{L}_{\mathcal{S}} : T(\mathcal{S}) \to H(\mathcal{S})$ is a homeomorphisms onto its image.

Let $s\beta$, $s > 0$, be an asymptotic ray to $Ł(T(\mathcal{S}))$ in $H(\mathcal{S})$, i.e. there exists a path $s \mapsto \alpha_s$ in $Ł(T(\mathcal{S}))$ such that the lifts $\tilde{\alpha}_s, \tilde{\beta}$ to the universal cover $\mathbf{D}\times\hat{G}$ satisfy $\|\frac{1}{s}\tilde{\alpha}_s - \tilde{\beta}\|_\nu \to 0$ as $s \to \infty$, for all $0 < \nu \leq 1$. This implies that $\|\tilde{\alpha}_s|_{\mathbf{D}\times\{t\}} - \tilde{\beta}|_{\mathbf{D}\times\{t\}}\|_\nu \to 0$ for all $t \in \hat{G}$ as $s \to \infty$. By [21], each $\tilde{\beta}|_{\mathbf{D}\times\{t\}}$, $t \in \hat{G}$, is a measured lamination on $\mathbf{D}\times\{t\}$. Therefore, $\beta \in H(\mathcal{S})$ is a leafwise measured lamination, and since it belongs to $H(\mathcal{S})$, it is continuous for the transverse variations. Thus β is a measured lamination on $\mathcal{S}$. The earthquake theorem (Theorem 5.1) for the universal hyperbolic solenoid $\mathcal{S}$ shows that an earthquake path $E_{s\mu} : \mathcal{S} \to X_s$,

$s>0$, is in $T(\mathcal{S})$. Denote by $\tilde{\mu}$ the lift of μ to the universal cover $\mathbf{D}\times\hat{G}$ of $\mathcal{S}$. Then $h_s:=E_{s\tilde{\mu}}$ is a leafwise earthquake map. By [21, Theorem 2], we have

$$\frac{1}{s}(h_s|_{\mathbf{D}\times\{t\}})_*(L)\to\tilde{\mu}|_{\mathbf{D}\times\{t\}},$$

in the ν-norm, $0<\nu\leq 1$, as $s\to\infty$ for each $t\in\hat{G}$. Moreover, [21, Lemma 4.4] shows that the above convergence in the ν-norm, $0<\nu\leq 1$, is uniform independent of the leaf $\mathbf{D}\times\{t\}$. Thus

$$(h_s)_*(L_{leaf})\to\mu,$$

as $s\to\infty$ in the Fréchet topology on $H(\mathcal{S})$. Thus the natural boundary to $T(\mathcal{S})$ (i.e. the space of asymptotic rays to Ł$(T(\mathcal{S}))$ in $H(\mathcal{S})$) is homeomorphic to the space $PML(\mathcal{S})$ of projective measured laminations on $\mathcal{S}$. The continuity of the action of the baseleaf preserving mapping class group on the closure $T(\mathcal{S})\cup PML(\mathcal{S})$ is immediate from [21]. □

Note that leafwise measured laminations on $\mathcal{S}$ are given in terms of continuity for the transverse variations. We show that each leafwise measured lamination on $\mathcal{S}$ is approximated by transversely locally constant (TLC) measured lamination (i.e. laminations obtained by lifting laminations on compact surfaces to $\mathcal{S}$) which is parallel to the statement that each hyperbolic metric on $\mathcal{S}$ is approximated by TLC hyperbolic metrics.

Theorem 6.3. *The subset of all measured lamination on the universal hyperbolic solenoid $\mathcal{S}$ which are locally transversely constant (TLC) is dense in the space of all measured laminations $ML(\mathcal{S})$ on $\mathcal{S}$ for the Fréchet topology.*

Proof. Recall that $T_{restr.}(\mathbf{D})$ is obtained by taking the closure of the union of all quasisymmetric maps which conjugate finite index subgroups of G onto other Fuchsian groups [14], i.e. the closure of the union $\cup_{[G:K]<\infty}\tilde{T}(\mathbf{D}/K)$ of the lifts $\tilde{T}(\mathbf{D}/K)$ to $T(\mathbf{D})$ of all Teichmüller spaces $T(\mathbf{D}/K)$ of finite degree unbranched covers $\mathbf{D}/K$ of $\mathbf{D}/G$. By the characterization of the image of Ł from [1], the image $\mathcal{L}(\cup_{[G:K]<\infty}\tilde{T}(\mathbf{D}/K))$ of the union consists of all bounded Hölder distributions α which are positive measures invariant under finite index subgroups of G and which satisfy $e^{-\alpha([a,b]\times[c,d])}+e^{-\alpha([b,c]\times[d,a])}=1$ for all $a,b,c,d\in S^1$ given in counterclockwise order.

Since Ł is a homeomorphism onto its image and Ł$(T(\mathbf{D}))$ is closed in $H(\mathbf{D})$ [21], it follows that Ł$(T_{restr.}(\mathbf{D}))$ equals to the closure (in the Fréchet topology) of Ł$(\cup_{[G:K]<\infty}\tilde{T}(\mathbf{D}/K))$. By [1] or [21], the asymptotic rays to Ł$(\cup_{[G:K]<\infty}\tilde{T}(\mathbf{D}/K))$ are containing all the projective measured laminations which are invariant under all finite index subgroups K of G. The Liouville map Ł composed with the projection $pr:H(\mathcal{S})\to S_\nu$ to the unit sphere S_ν in $H(\mathcal{S})$ (for a fixed ν-norm) is a homeomorphism (see [21]). This implies that the points in the closure of $pr\circ$ Ł$(T_{restr.}(\mathbf{D}))$ which are not in $pr\circ$Ł$(T_{restr.}(\mathbf{D}))$ are projectivized asymptotic rays to Ł$(T_{restr.}(\mathbf{D})$. Any such point in the closure of $pr\circ$ Ł$(T_{restr.}(\mathbf{D}))$ is approximated by projective measures in $pr\circ$Ł$(T_{restr.}(\mathbf{D}))$ that are invariant under finite index subgroups of G. Thus the closure of all invariant (under finite index subgroups of G) projective measured laminations contains all asymptotic rays to Ł$(T_{restr.})$. The measured laminations invariant under finite index subgroups of G lift to locally transversely constant measured laminations on $\mathcal{S}$. Thus the limits of locally transversely constant measured laminations on $\mathcal{S}$ give all (transversely continuous) measured laminations on $\mathcal{S}$. □

Remark 6.4. The set of asymptotic rays to $Ł(T_{restr.}(\mathbf{D}))$ in $H(\mathbf{D})$ is equal to the restriction to the baseleaf of the set of asymptotic rays to $Ł(T(\mathcal{S}))$ in $H(\mathcal{S})$. This is a consequence of the proof of Theorem 6.3, since each asymptotic ray for $Ł(T_{restr.}(\mathbf{D}))$ is the limit in the Fréchet topology of the asymptotic rays invariant under finite index subgroups of G.

7. The punctured solenoid

We sketch an extension of our results to the punctured solenoid $\mathcal{S}_p$ defined in [17]. We first recall the definition of $\mathcal{S}_p$.

Let H be the subgroup of $PSL_2(\mathbb{Z})$ such that $\mathbf{D}/H$ is a once punctured torus. Denote by $\hat{H}$ the profinite completion of H. Then we define (see [17]) the punctured solenoid by

$$\mathcal{S}_p := (\mathbf{D} \times \hat{H})/H,$$

where the action of H is given by $A(z,t) := (A(z), tA^{-1})$, for $A \in H$.

A leafwise measured lamination on $\mathcal{S}_p$ is an assignment of a bounded measured lamination to each leaf of $\mathcal{S}_p$ which varies continuously in the transverse direction. The support of a leafwise measured lamination on $\mathcal{S}_p$ is a leafwise geodesic lamination which is not necessarily continuous for the transverse variations. We say that the support of a leafwise measured lamination on $\mathcal{S}_p$ is compact if, when restricted to each leaf of $\mathcal{S}_p$, the support geodesic lamination is a pre-compact subset of $\mathcal{S}_p$. The earthquake theorem holds for $\mathcal{S}_p$ when we use the space of measured laminations with compact support $ML_0(\mathcal{S}_p)$.

If a simple geodesic on a punctured surface enters a definite neighborhood of a puncture, then it has an endpoint at the puncture. We recall a standard proof of this fact in the upper half-plane model of the hyperbolic plane. Let $A : z \mapsto z+1$ be the parabolic element corresponding to the puncture. If a lift of the geodesic entering a neighborhood of a puncture on the surface is a Euclidean half-circle with radius greater than $1/2$, then the translate of the lift of the geodesic by A intersects the lift of the geodesic. Thus the geodesic is not simple. Contradiction. Therefore, there exists a definite neighborhood of a puncture where no simple geodesic enters unless it ends at the puncture.

Assume that a measured lamination μ on $\mathcal{S}_p$ does not have a compact support. Then there exists a leaf l of $\mathcal{S}_p$ such that the restriction μ_l of μ contains a geodesic $g \in l$ with an endpoint at the puncture. By the continuity for the transverse variations, the support of μ_l contains the translates of g in l by parabolic elements $A^{kn} \in H$, for a fixed $k > 0$ and for all $n \in \mathbb{N}$, where A fixes the endpoint of g. Then g has to be isolated in l because otherwise the translates under A^{kn} of the geodesics in the support of μ_l converging to g would intersect the geodesics converging to g similar to the punctured surface case. Thus μ_l would not be a measured lamination. The remaining possibility is that g is isolated with atomic measure. Then the translates $A^{kn}(g)$ also have atomic measure approximately equal to the atomic measure of g for n large and they share the same endpoint. Then the measured lamination $\mu|_l$ is not bounded which is a contradiction with the definition of a measured lamination on $\mathcal{S}_p$. Therefore, a measured lamination μ on $\mathcal{S}_p$ always has a compact support.

As in the proof of Theorem 5.1 it follows that the extension to the boundary of each quasiconformal map $f : \mathcal{S}_p \to X_p$ can be achieved by a leafwise earthquakes whose measures vary continuously in the transverse direction for the Fréchet topology on earthquake measures. On the other hand, a measured lamination μ on $\mathcal{S}_p$ with compact support gives a quasiconformal map $f : \mathcal{S}_p \to X_p$ whose extensions to the boundaries of the leaves agrees with the earthquake E_μ. Thus the earthquake theorem is true for the punctured solenoid for the space of measured laminations $ML_0(\mathcal{S}_p)$ with the compact support on $\mathcal{S}_p$.

We explain how to extend Thurston's boundary to the Teichmüller space of the punctured solenoid $\mathcal{S}_p$. We show that the boundary consists of $PML_0(\mathcal{S}_p)$. By the extension of the earthquake theorem and by the proof of Theorem 6.2, it is only necessary to show that the asymptotic rays to $\L(T(\mathcal{S}_p)) \subset H(\mathcal{S}_p)$ are of the form tW, $t > 0$, for some $W \in ML_0(\mathcal{S}_p)$. Let $\alpha_t \in \L(T(\mathcal{S}_p))$ be such that $\frac{1}{t}\alpha_t \to W$ as $t \to \infty$. Then the restriction of W to each leaf of $\mathcal{S}_p$ is a bounded measured lamination by the results in [21]. Since W is continuous for the transverse variations, it follows that W is a measured lamination on $\mathcal{S}_p$. By the above, $W \in ML_0(\mathcal{S}_p)$.

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